KVS Junior Mathematics Olympiad (JMO)

SAMPLE PAPER – 5

M.M. 100 Time : 3 hours

Note : Attempt all questions.
All questions carry equal marks

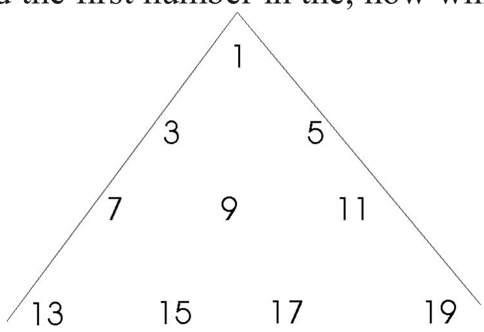
Q. 1 Resolve into factors : $a^4(b-c)+b^4(c-a)+c^4(a-b)$

Q.2 Simplify : $\frac{x-y}{(a+x)(a+y)} + \frac{y-z}{(a+y)(a+z)} + \frac{z-x}{(a+z)(a+x)}$

Q.3. Two circles with radii a and b touch each other externally. Let c be the radius of a circle which touches these two circles as well as a common tangent to two circles prove that $\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$

Q 4. Find the unit digit of 777^{777}

Q 5. The odd numbers are arranged in a triangular pattern as shown.
If this pattern continues find the first number in the, now which has a sum of 1000000.



Q.6 In 1932, I was as old as the last two digits of my birth year. My grandfather said that that applies to him also. How old are we ?

Q. 7 A hollow cone is cut by a plane parallel to the base and the upper portion is removed. If the curved surface of the remainder is $\frac{8}{9}$ of the curved surface of the whole cone. Find the ratio of the line segments into which the conds attitude in divided by the place.

Q.8 In triangle XYZ, W is the point at intersection of XY with interior bisector of angle Z, and V in the mid point of XY. Prove that ZW + ZV < XZ + YZ.

Q.9 Show that: $1^4+2^4+3^4+4^4+\ ..\ +\eta^4 = \frac{\eta}{30}\left[6n^4+5\eta^3+10\eta^2-1\right]$

Q.10 Five letters are written to five different persons and their addresses are on 5 envelopes. Find in now many ways letters can be placed in envelopes so that no letter is placed in the correct envelope.

SOLUTION AND HINTS (SAMPLE PAPER 5)

Q. 1 $a^4(b-c)+b^4(c-a)+c^4(a-b)$

$= (b-c)\left\{a^4+bc\left(b^2+bc+c^2\right)-a(b+c)\left(b^2+c^2\right)\right\}$

$= (b-c)\left\{a^4-a\left(b^3+b^2c+bc^2+c^2\right)+\text{bc}\left(\text{b}^2+bc+c^2\right)\right\}$

$= (b-c)\left\{-b^3(a-c)-b^2c(a-c)-bc^2(a-c)+a\left(a^3-c^3\right)\right\}$

$= (b-c)(a-c)\left\{-b^3-b^2c-bc^2+a\left(a^2+ac+c^2\right)\right\}$

$= (b-c)(a-c)\left\{\text{c}^2(a-b)+c\left(a^2-b^2\right)+\left(a^3-b^3\right)\right\}$

$= (b-c)(a-c)(\text{a - b})\left\{c^2+c(a+b)+a^2+ab+b^2\right\}$

$= (b-c)(a-c)(\text{a - b})\left(\text{a}^2+b^2+c^2+ab+bc+ca\right)$

Q. 2 LCM of the denominator

$= (a+x)(a+y)(a+z)$

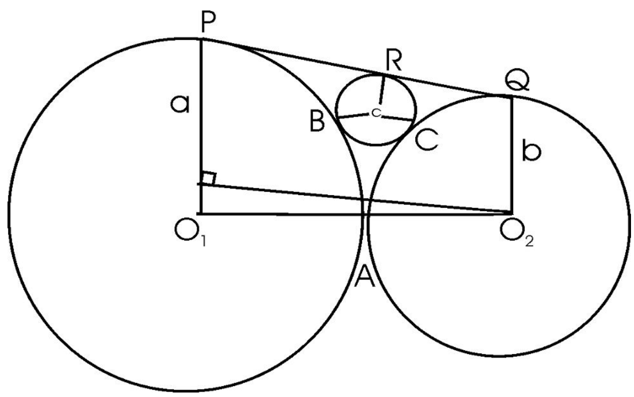
so, the given expression

$= \dfrac{(a+z)(x-y)+(a+x)(y-z)+(a+y)(z-x)}{(a+x)(a+y)(a+z)}$

$= \dfrac{0}{(a+x)(a+y)(a+z)}$

$= 0$

Q. 3



In the figure two circles touch externally a point A with centers be at O_1 and O_2. PQ is common tangent and circle of radius C touches the two circles at B and C respectively.

PQ is tangent at R.

A > b

Let us draw perpendicular O_2S on O_1P. Then PQO_2S is a rectangle and

PQ = O_2S = $\sqrt{0_1 0_2^2 - 0_1 S^2}$ = $\sqrt{(a+b)^2 - (a-b)^2}$ = $2\sqrt{ab}$

Similarly, PR = $2\sqrt{ac}$, RQ = $2\sqrt{bc}$

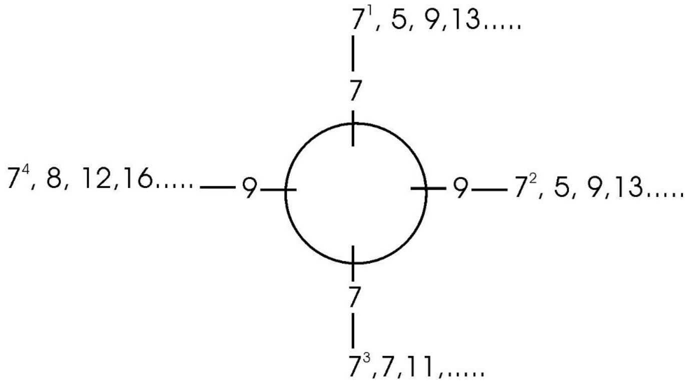
Now, PQ = PR + RQ

$\Rightarrow \quad 2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}$

$\Rightarrow \quad \frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$

Q.4.

Expression	Unit digit
7^1	7
7^2	9
7^3	3
7^4	1
7^5	7
7^6	9
7^7	3
7^8	1
7^9	7
7^{10}	9
7^{11}	3



Cycle of unit digit rotation

We know that

776 = 4x 194

So unit digit for 777^{776} is 1.

Q. 5 Let us tabulate the sum and first numbers in row.

Row	Sum	First number in the row
1	$1=1^3$	1=1x0+1
2	$8=2^3$	3=2x1+1
3	$27=3^3$	7=3x2+1
4	$64=4^3$	13=4x3+1
..	..	..
100	$1000000=100^3$	9901=100x99+1

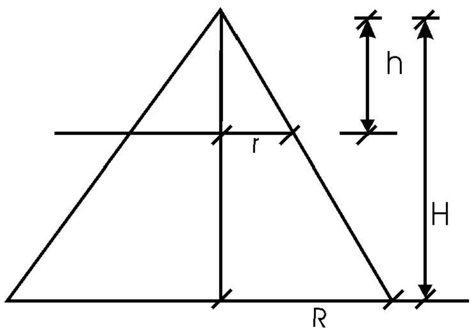
So, first number = 9901

Q. 6 By simple comparison of age with year.

Year	Age	Year	Age	Year	Age
1932	0	1909	23	1882	50
1931	1	1908	24	1881	51
1930	2	1907	25	1880	52
1929	3	1906	26	1879	53
1928	4	1905	27	1878	54
1927	5	1904	28	1877	55
1926	6	1903	29	1876	56
1925	7	1902	30	1875	57
1924	8	1901	31	1874	58
1923	9	1900	32	1873	59
1922	10	1899	33	1872	60
1921	11	1898	34	1871	61
1920	12	1897	35	1870	62
1919	13	1896	36	1869	63
1918	14	1895	37	1868	64
1917	15	1894	38	1867	65
*1916	16	1983	39	*1866	66
1915	17	1892	40	1865	67
1914	18	1891	41	1864	68
1913	19	1890	42	1863	69
1912	20	1889	43	1862	70
1911	21	1888	44	1861	71
1910	22	1887	45	1860	72
		1886	46	1859	73
		1885	47	1858	74
		1884	48	1857	75
		1883	49	1856	76
				1855	77

We find that in 1916 the age is 16 and for grand father in 1996 the age is 66.

Q.7

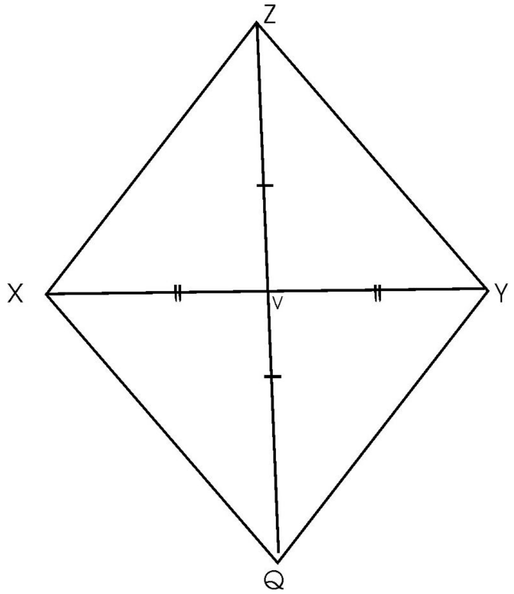


From the figure

We have

$$\frac{h}{R} = \frac{r}{R}$$

For second case,



ZV is median through $\angle z$

In ΔZYU and ΔXVQ

Two sides and between angle in equal, hence triangles are congruent.

So, XQ = ZY

XQ + XZ > ZQ

$\Rightarrow xz + zy > 2zv$ (II)

now adding (i) and (ii)

xz +zy > 2 ZW

XZ + ZY > 2ZV

$\Rightarrow \quad 2xz + 2 = Y > 2ZW + 2ZV$

$\Rightarrow$ XZ + ZY > ZW + ZV

Q 8 We know :

$n^5 - (n-1)^5 = 5n^4 - 10n^3 + n^3 + 10n^2 - 5n + 1$

On putting n=1,2,3,4,… ,, we get

$1^5 - 0^5 = 5.1^4 - 10.1^3 + 10.1^2 - 5.1 + 1$

$2^5 - 1^5 = 5.2^4 - 10.2^3 + 10.2^2 - 5.2 + 1$

$3^5 - 2^5 = 5.3^4 - 10.3^3 + 10.3^2 - 5.3 + 1$

$4^5 - 3^5 = 5.3^4 - 10.4^3 + 10.3^2 - 5.4 + 1$

… … … … … ... …

$n^5 - (n-1)^5 = 5n^4 - 10n^3 + 10n^2 - 5 \cdot n + 1$

Adding

The curved surface area of smaller cone = $r\pi\,\lambda$

And for remaining truncated figure

$= (\pi\, R\, H - \pi\, r\, h)$

form the question

$\pi\ R\,H - \pi\ r\,h = \frac{8}{9}\pi$ R H

$\Rightarrow \quad \frac{1}{9}\pi \ R H = \pi r h$

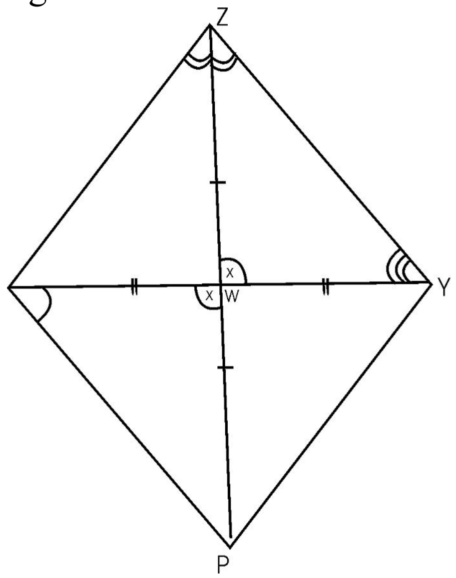
We have $\quad \frac{r}{R} = \frac{h}{H}$

So

$$\frac{1}{9} \pi = \pi \frac{h}{H^2}$$

$\Rightarrow \quad \frac{h}{H} = \frac{1}{3}$

Q.9 Taking ZW as angle bisector for $\angle$Z



In Δ XPW & ΔZWY

WZ =WP

$\angle$ZWY = $\angle$XWP

and $\angle$WXP = $\angle$WYZ

So ΔXPW $\cong$ WZY (congruent)

$\therefore$ XW = WY

XZ + XP > ZP

$\Rightarrow$ XZ + ZY > 2ZW ……(i)

$\Rightarrow \quad n^5 = 5\sum n^4 - 10\sum n^3 + 10\sum n^2 - 5\sum n + n$

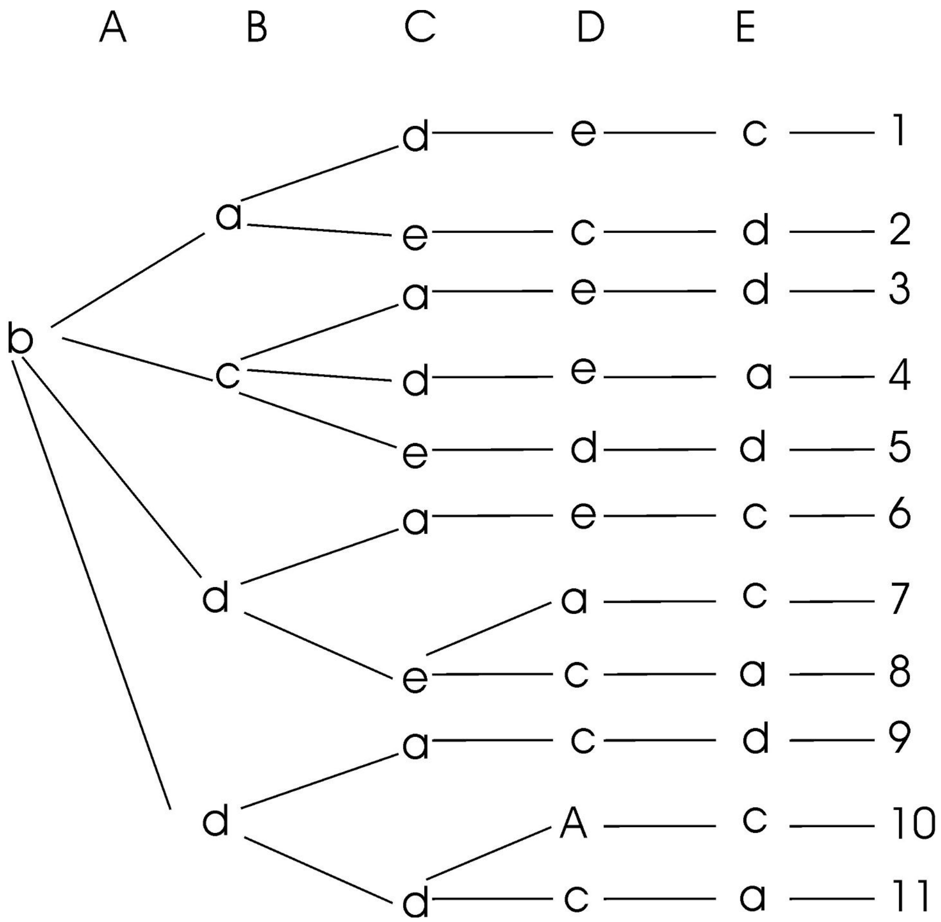
$\Rightarrow \quad n^5 = 5\sum n^4 - 10 \left\{\frac{n(n+1)}{2}\right\}^2 + 10 \frac{n(n+1)(2n+1)}{6} - 5 \frac{n(n+1)}{2} + n$

$\Rightarrow \quad 5\sum n^4 = n\left[n^4 + \frac{5}{2} \{n(n+1)\}^2 - \frac{5}{2} (n+1) (2n+1) + \frac{5}{2} (n+1) - 1\right]$

$\Rightarrow \quad \sum n^4 = \frac{n}{30}\left[6n^4 + 15n \left(n^2 + 2n + 1\right) - 10\left(2n^2 + 3n + 1\right) + 15(n+1) - 6 \right]$

$= \quad \frac{n}{30}\left[6n^4 + 15n^3 + 10n^2 - 1\right]$

Q. 10 Let's mae a table as below. If letters are represented by a,b,c,d, and e. The addresses by A, B, C, D and E.



Similarly if c is sent to A there will be 11 ways. Hence total - 4x11 = 44 ways.