Chapter 1

Algebra

1.1 Polynomial Functions

Any function $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ is a polynomial function if $a_i (i = 0, 1, 2, 3, ..., n)$ is a constant which belongs to the set of real numbers and the indices, $n, n-1, ..., 1$ are natural numbers. If $a_n \neq 0,$ then we say that $f(x)$ is a polynomial of degree *r*.

Example 1. $x^4 - x^3 + x^2 - 2x + 1$ is a polynomial of degree 4 and 1 is a zero of the polynomial as $1^4 - 1^3 + 1^2 - 2 \times 1 + 1 = 0.$

Also,

2. $x^3 - ix^2 + ix + 1 = 0$ is a polynomial of degree 3 and i is a zero of his polynomial as $i^3 - i.i^2 + i.i + 1 = -i + i - 1 + 1 = 0.$

Again,

3. $x^2 - (\sqrt{3} - \sqrt{2})x - \sqrt{6}$ is a polynomial of degree 2 and $\sqrt{3}$ is a zero of this polynomial as $(\sqrt{3})^2 - (\sqrt{3} - \sqrt{2})\sqrt{3} - \sqrt{6} = 3 - 3 + \sqrt{6} - \sqrt{6} = 0.$

Note : The above definition and examples refer to polynomial functions in one variable. similarly polynomials in 2, 3, ..., *n* variables can be defined, the domain for polynomial in *n* variables being set of (ordered) *n* tuples of complex numbers and the range is the set of complex numbers.

Example : $f(x, y, z) = x^2 - xy + z + 5$ is a polynomial in x, y, z of degree 2 as both x^2 and *xy* have degree 2 each.

Note : In a polynomial in *n* variables say $x_1, x_2, ..., x_n,$ a general term is $x_1^{k_1}, x_2^{k_2}, ..., x_n^{k_n}$ where degree is $k_1 + k_2 + ... + k_n$ where $k_i \geq 0, i = 1, 2, ..., n.$ The degree of a polynomial in *n* variables is the maximum of the degrees of its terms.

Division in Polynomials

If $P(x)$ and $\phi(x)$ are any two polynomials then we can find polynomials $Q(x)$ and $R(x)$ such hat $P(x) = \phi(x) \times Q(x) + R(x)$ where the degree of $R(x) <$ degree of $Q(x).$

$Q(x)$ is called the quotient and $R(x),$ the remainder.

In particular if $P(x)$ is a polynomial with complex coefficients and a is a complex number then there exists a polynomial $Q(x)$ of degree 1 less than $P(x)$ and a complex number R, such that

$$P(x) = (x-a)Q(x) + R.$$

Example : $$x^5 = (x-a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4) + a^5$$

Here $P(x) = x^5,$

$Q(x) = x^4 + ax^3 + a^2x^2 + a^3x + a^4$

and $R = a^5.$

1.3. Remainder Theorem and Factor Theorem

Reminder Theorem : If a polynomial $f(x)$ is divided by $(x-a)$ then the remainder is equal to $f(a).$

Proof : $f(x) = (x-a)Q(x) + R$ …(1)

and so $f(a) = (a-a)Q(a) + R = R$

If $R = 0$ then $f(x) = (x-a)Q(x)$ and hence $(x-a)$ is a factor of $f(x).$

Further $f(a) = 0$ and thus a is a zero of the polynomial $f(x).$ This leads to the factor theorem.

Factor Theorem : $(x-a)$ is a factor of polynomial $f(x)$ if and only if $f(a) = 0.$

Fundamental theorem of algebra : Every polynomial function of degree ≥ 1 has at least one zero in the complex numbers. In other words if we have

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$$

with $n \geq 1,$ then there exists at least one $h \in C$ such that,

$$a_nh^n + a_{n-1}h^{n-1} + ... + a_1h + a_0 = 0$$

From this it is easy to deduce that a polynomial function of degree '*n*' has exactly *n* zeroes.

Example 1. Find the polynomial function of lowest degree with integral coefficient s with $\sqrt{5}$ as one of its zeroes.

Solution : Since the order of the surd $\sqrt{5}$ is 2, you can expect the polynomial of the lowest degree to be a polynomial of degree 2.

Let $$P(x) = ax^2 + bx + c;\, a, b, c \in Q$$

$$P(\sqrt{5}) = 5a + \sqrt{5}b + c = 0$$

But $\sqrt{5}$ is a zero, so $5a + c = 0$ and $\sqrt{5}b = 0$

$\Rightarrow$ $c = -5a$ and $b = 0.$

So the required polynomial function is $P(x) = ax^2 - 5x.$

You can find the other zero of this polynomial to *i.e.*, $-\sqrt{5}.$

Example 2. If *x, y, z* be positive numbers, show that

$$(x+y+z)^3 \geq 27xyz.$$

Solution : Since A.M. (arithmetic mean) $\geq$ G.M. (geometric men), therefore

$$\frac{x+y+z}{3} \geq (xyz)^{1/3}.$$

Cubing both sides and multiplying throughout by 27, we have

$$(x+y+z)^3 \geq 27xyz.$$

Example 3. If $a_1, ..., a_n, b_1, ..., b_n$ be real numbers and none of the b_i's be zero, then prove that

$$(a_1^2+...+a_n^2)(b_1^{-2}+...+b_n^{-2}) \geq (a_1/b_1+...+a_n/b_n)^2.$$

Solution : Applying Cauchy-Schwarz inequality to the numbers $a_1, ..., a_n, b_1^{-1}, ..., b_n^{-1},$ we have

$$\left\{a_1\left(\frac{1}{b_1}\right)+...+a_n\left(\frac{1}{b_n}\right)\right\}^2 \leq (a_1^2+..+a_n^2)(b_1^{-2}+...+b_n^{-2}),$$

or $$(a_1^2+...+a_n^2)(b_1^{-2}+...+b_n^{-2}) \geq \left(\frac{a_1}{b_1}+...+\frac{a_n}{b_n}\right)^2.$$

Example 4. (Triangle Inequality). If $x_1, x_2, y_1, y_2,$ be any real numbers, then show that

$$\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2\}} \leq \sqrt{(x_1^2+y_1^2)}+\sqrt{(x_2^2+y_2^2)},$$

where the sign $\sqrt{\ }$ denotes the positive square root.

Solution : $(x_1-x_2)^2+(y_1-y_2)^2 = (x_1^2+y_1^2)+(x_2^2+y_2^2)-2(x_1x_2+y_1y_2).$ …(i)

By Cauchy-Schwarz inequality,

$$(x_1x_2+y_1y_2)^2 \leq (x_1^2+y_1^2)(x_2^2+y_2^2),$$

i.e., $$|x_1x_2+y_1y_2| \leq \sqrt{(x_1^2+y_1^2)}\sqrt{(x_2^2+y_2^2)}$$ …(ii)

From (i) and (ii), we have

$$(x_1-x_2)^2+(y_1-y_2)^2 \leq (x_1^2+y_1^2)+(x_2^2+y_2^2)+2\sqrt{x_1^2+y_1^2}\sqrt{(x_2^2+y_2^2)},$$

or $$(x_1-x_2)^2+(y_1-y_2)^2 \leq \{\sqrt{(x_1^2+y_1^2)}+\sqrt{(x_2^2+y_2^2)}\}^2$$

Taking positive square roots, we have

$$\sqrt{\{(x_1-x_2)^2+(y_1-y_2)^2\}} \leq \sqrt{(x_1^2+y_1^2)}+\sqrt{(x_2^2+y_2^2)}.$$

Remark : Geometrically interpreted, the above inequality expresses the fact the sum of two sides of a triangle can never be less than the third side and this is precisely the reason for the name 'triangle inequality'.

Example 5. If $c_1, ..., c_n$ be positive real numbers, show that

$$(c_1 + ... + c_n)^3 \le n^2 (c_1^3 + ... + c_n^3).$$

When does the inequality reduce to equality ?

Solution : If $a_1, ..., a_n, b_1, ..., b_n,$ be real numbers, then by Cauchy-Schwarz inequality,

$$(a_1 b_1 + ... + a_n b_n)^2 \le (a_1^2 + ... + a_n^2)9b_1^2 + ... + b_n^2). \qquad ...(1)$$

Putting $a_i = c_i^{3/2}, b_i = c_i^{1/2}, (i = 1, 2, ..., n)$ in the above inequality, we have

$$(c_1^2 + ... + c_n^2)^2 \le (c_1^3 + .. + c_n^3)(c_1 + ... + c_n). \qquad ...(2)$$

Again, putting $a_i = c_i, b_i = 1, (i = 1, 2, ..., n)$ in (1), we have

$$(c_1 + ... + c_n)^2 \le n(c_1^2 + ... + c_n^2). \qquad ...(3)$$

Squaring both sides of (3) and using (2), we immediately have

$(c_1 + ... + c_n)^3 \le n(c_1^3 + ... + c_n^3).$

The above inequality reduces to an equality iff each of the inequalities (2) and (3) reduces to an equality, *i.e.*, iff

$$c_1^{3/2} : ... : c_n^{3/2} :: c_1^{1/2} : ... : c_n^{1/2},$$

and $\quad c_1 : ... : c_n = 1 : ... : 1,$

i.e., iff $\quad c_1 = c_2 = ... = c_n.$

Example 6. If x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 27,$ then show that $x^3 + y^3 + z^3 \ge 81.$

Solution : Applying Cauchy-Schwarz inequality to the two sets of numbers

$$x^{3/2}, y^{3/2}, z^{3/2}; x^{1/2}, y^{1/2}, z^{1/2}$$

we have $\quad (x^2 + y^2 + z^2)^2 \le (x^3 + y^3 + z^3)(x + y + z). \qquad ...(i)$

Again, applying Cauchy-Schwarz inequality to the two sets of numbers

$$x, y, z; 1, 1, 1$$

we have $\quad (x + y + z)^2 \le 3(x^2 + y^2 + z^2) \qquad ...(ii)$

Squaring both sides of (i), we have

$$(x^2 + y^2 + z^2)^4 \le (x^3 + y^3 + z^3)^2 (x^3 + y^3 + z^3) \qquad ...(iii)$$

i.e., $\quad (x^2 + y^2 + z^2)^4 \le (x^3 + y^3 + z^3)^2 (x^2 + y^2 + z^2)$

Since $x^2 + y^2 + z^2 = 27,$ we have from (iii)

$$(x^3 + y^3 + z^3)^2 \ge (81)^2.$$

Taking positive square roots, we have

$$x^3 + y^3 + z^3 \geq 81.$$

Tcheby Chef's Inequality

Example 7. If $a_1, a_2, a_3, b_1, b_2, b_3$ are any real numbers such that $a_1 \leq a_2 \leq a_3, b_1 \leq b_2 \leq b_3$, then show that

$$3(a_1b_1 + a_2b_2 + a_3b_3) \geq (a_1 + a_2 + a_3)(b_1 + b_2 + b_3).$$

Solution : Since $a_1 \leq a_2, b_1 \leq b_2$, therefore, $a_1 - a_2, b_1 - b_2$ are of the same sign or at least one of them is zero, so that

$(a_1 - a_2)(b_1 - b_3) > 0,$ and therefore

$$a_1b_1 + a_2b_2 \geq a_1b_2 + a_2b_1.$$

Similarly, $$a_2b_2 + a_3b_3 \geq a_2b_3 + a_3b_2, \quad \ldots\text{(ii)}$$

and $$a_3b_3 + a_1b_1 \geq a_3b_1 + a_1b_3. \quad \ldots\text{(iii)}$$

Adding (i), (ii) and (iii) and then adding $a_1b_1 + a_2b_2 + a_3b_3$ to both sides of resulting inequality, we have

$$3(a_1b_1 + a_2b_2 + a_3b_3) \leq (a_1 + a_2 + a_3)(b_1 + b_2 + b_3).$$

Theorem : If $a_1, ..., a_n$ and $b_1, ..., b_n$ are any real numbers, such that

(i) $a_1 \leq ... \leq a_n, b_1 \leq ... \leq b_n$, then

$$n(a_1b_1 + ... + a_nb_n) \geq (a_1 + ... + a_n)(b_1 + ... + b_n).$$

(ii) $a_1 \geq ... \geq a_n, b_1 \leq ... \leq b_n$, then

$$n(a_1b_1 + ... + a_nb_n) \leq (a_1 + .. + a_n)(b_1 + ... + b_n).$$

Proof : (i) For every pair of distinct suffixes *p* and *q*, the differences $a_p - a_q$ and $b_p - b_q$ are of the same sign or at least one of them is zero.

Hence, $(a_p - a_q)(b_p - b_q) \geq 0$

i.e., $d_pb_p + a_qb_q \geq a_pb_q + a_qb_p.$

There are $\frac{1}{2}n(n-1)$ inequalities of the above type (for there are $\frac{1}{2}n(n-1)$ pairs of distinct suffixes *p*, *q*), Adding the corresponding sides of all such inequalities, we obtain

$$(n-1)(a_1b_1 + ... + a_nb_n) \geq (a_1 + ... + a_n)(b_1 + ... + b_n) - (a_1b_1 + .. + a_nb_n)$$

i.e., $$n(a_1b_1 + ... + a_nb_n) \geq (a_1 + ... + a_n)(b_1 + ... + b_n).$$

(ii) For every pair of distinct suffixes *p* and *q*, $a_p - a_q$ and $b_p - b_q$ are of opposite signs or at least one of them is zero. Hence,

$$(a_p - a_q)(b_p - b_q) \le 0$$

i.e., $$a_p b_p + a_q b_q \le a_p b_q + a_q b_p.$$

Adding the corresponding sides of all the $\frac{1}{2}n(n-1)$ inequalities of the above type, we obtain

$$(n-1)(a_1b_1 + \ldots + a_nb_n) \le (a_1 + \ldots + a_n)(b_1 + \ldots + b_n) - (a_1b_1 + \ldots + a_nb_n),$$

i.e., $$n(a_1b_1 + \ldots + a_nb_n) \le (a_1 + \ldots + a_n)(b_1 + \ldots + b_n).$$

Remark : The inequality above can be put in the following symmetric form :

$$\frac{a_1b_1 + \ldots + a_nb_n}{n} \ge \frac{a_1 + \ldots + a_n}{n} \cdot \frac{b_1 + \ldots + b_n}{n}.$$

This form suggests the following generalisation which we state without proof.

If $a_1, \ldots, a_n; b_1, \ldots, b_n; \ldots; k_1, \ldots, k_n$ are real numbers such that

$$a_1 \le \ldots \le a_n,\ b_1 \le \ldots \le b_n,\ \ldots,\ k_1 \le \ldots \le k_n,$$

then, $$\frac{a_1b_1\ldots k_1 + \ldots + a_nb_n\ldots k_n}{n} \ge \frac{a_1 + \ldots + a_n}{n} \cdot \frac{b_1 + \ldots + b_n}{n} \ldots \frac{k_1 + \ldots + k_n}{n}$$

We shall refer to this inequality as Generalised Tchebychef's Inequality.

Example 8. Show that :

(a) $$\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n} \le n\sqrt{\left\{\frac{(n+1)}{2}\right\}};$$

(b) $$\left(1 + \sqrt{\frac{1}{2}} + \ldots + \sqrt{\frac{1}{n}}\right) / \sqrt{n} \le (2n-1)^{1/4}$$

Solution : (a) Applying Tchebychef's inequality to the sets of numbers $\sqrt{1}, \ldots, \sqrt{n}; \sqrt{1}, \ldots, \sqrt{n}$, we have

$$n(\sqrt{1}.\sqrt{1} + \sqrt{2}.\sqrt{2} + \ldots + \sqrt{n}\sqrt{n}) \ge (\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n})^2,$$

or $$n(1 + 2 + \ldots + n) \ge (\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n})^2,$$

or $$n^2 \frac{(n+1)}{2} \ge (\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n})^2.$$

Therefore, $$\sqrt{1} + \sqrt{2} + \ldots. + \sqrt{n} \le n\sqrt{\left\{\frac{(n+1)}{2}\right\}}.$$

(b) Applying Tchebychef's inequality to the sets of numbers $1, \frac{1}{2}, \ldots, \frac{1}{n}; 1, \frac{1}{2}, \ldots, \frac{1}{n}$, we obtain

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)^2 \le n\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right),$$

$$\le n\left[1+\frac{1}{1.2}+\ldots+\frac{1}{(n-1)n}\right].$$

$$= n\left[1+\left(1-\frac{1}{2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)\right],$$

$$= n\left(1+1-\frac{1}{n}\right).$$

Taking positive square roots of both sides, we have

$$\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right) \le \sqrt{(2n-1)} \qquad \ldots\text{(i)}$$

Again, applying Tchebychef's inequality to the sets of numbers $1, \sqrt{\frac{1}{2}}, \ldots, \sqrt{\frac{1}{n}} : 1, \sqrt{\frac{1}{2}}, \ldots, \sqrt{\frac{1}{n}},$ we have

$$\left(1+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)^2 \le n\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right). \qquad \ldots\text{(ii)}$$

From (i) and (ii), we have

$$\left\{1+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\left(\frac{1}{n}\right)}\right\}^2 \le n\sqrt{(2n-1)}.$$

Therefore $$\frac{\left(1+\sqrt{\frac{1}{2}}+\ldots+\sqrt{\frac{1}{n}}\right)}{\sqrt{n}} \le (2n-1)^{1/4}$$

Example 9. If a, b, c are all positive and no two of them are equal, then prove that

(a) $a^3+b^3+c^3 > \frac{(a+b+c)^3}{9} > 3abc.$

(b) $a^4+b^4+c^4 > abc(a+b+c).$

Solution : (a) Without any loss of generality we may assume that $a < b < c$. By applying the generalised Tchebychef's inequality to three sets of numbers each of which is the same as $a, b, c,$ we obtain

$$\frac{a^3+b^3+c^3}{3} > \frac{a+b+c}{3}.\frac{a+b+c}{3}.\frac{a+b+c}{3},$$

i.e., $$a^3+b^3+c^3 > \frac{(a+b+c)^3}{9} \qquad \ldots\text{(i)}$$

Again, since the arithmetic mean exceeds the geometric mean

$$\left(\frac{a+b+c}{3}\right)^3 > abc \qquad \ldots\text{(ii)}$$

From (i) and (ii), we obtain the inequalities

$$a^3+b^3+c^3 > \frac{(a+b+c)}{9} > 3abc. \qquad \ldots\text{(a)}$$

(b) As in (a), without any loss of generality we may assume that $a<b<c.$ Since $a<b<c,$ therefore, $a^3<b^3<c^3.$

Applying Tchebychef's inequality to the sets of numbers $a, b, c; a^3, b^3, c^3,$ we obtain

$$\frac{a^4+b^4+b^4}{3} > \frac{a^3+b^3+c^3}{3}.\frac{a+b+c}{3} \qquad \ldots\text{(iii)}$$

Also, from (a) $\dfrac{a^3+b^3+c^3}{3} > abc.$ …(iv)

From (iii) and (iv), we have

$$a^4+b^4+c^4 > abc(a+b+c).$$

Example 10. If a, b, c are positive and unequal, show that

$$(a^7+b^7+c^7)(a^2+b^2+c^2) > (a^5+b^5+c^5)(a^4+b^4+c^4),$$

Solution : $(a^7+b^7+c^7)(a^2+b^2+c^2)-(a^5+b^5+c^5)(a^4+b^4+c^4),$

$$= \Sigma(a^7b^2+a^2b^7-a^5b^4-a^4b^5),$$

$$= \Sigma a^2b^2(a^5+b^5-a^3b^2-a^2b^3),$$

$$= \Sigma a^2b^2(a^3-b^3)(a^2-b^2).$$

The differences a^2-b^2, a^3-b^3 are both of the same sign, and therefore, $(a^2-b^2)(a^3-b^3)$ is positive. Similarly, the other two terms in the above sum are also positive. Therefore,

$$(a^7+b^7+c^7)(a^2+b^2+c^2)-(a^5+b^5+c^5)(a^4+b^4+c^4) > 0.$$

Example 11. If a, b, c are positive and if p, q, r are rational numbers such that $p-q-r(\neq 0)$ and $r(\neq 0)$ have the same sign, then show that

$$(a^p+b^p+c^p)(a^q+b^q+c^q) \geq (a^{p-r}+b^{p-r}+c^{p-r})(a^{q+r}+b^{q+r}+c^{q+r}).$$

Show that if either

(i) $a=b=c,$ or (ii) $p=q+r,$ or (iii) $r=0,$ then equality holds.

Solution : $(a^p+b^p+c^p)(a^q+b^q+c^q)-(a^{p-r}+b^{p-r}+c^{p-r})(a^{q+r}+b^{q+r}+c^{q+r}).$

$$= \Sigma(b^qa^p+a^qb^p-a^{p-r}b^{q+r}-a^{q+r}c^{p+r}),$$

$$= \Sigma a^qb^q(a^{p-q}+b^{p-q}-a^{p-q-r}b^r-a^rb^{p-q-r}),$$

$$= \Sigma a^q b^q (a^{p-q-r} - b^{p-q-r})(a^r - b^r).$$

Since $p - q - r$ and r have the same sign, the differences $a^{p-q-r} - b^{p-q-r}$ and $a^r - b^r$ have the same sign or are both zero.

Therefore,

$$a^q b^q (a^{p-q-r} - b^{p-q-r})(a^r - b^r) \geq 0,$$

and similarly each of the other two terms in the above sum is also non-negative, so that the sum is non-negative. This proves the inequality.

Also, if any of the given conditions is satisfied, then at least one of the factors in each term in $\Sigma a^q b^q (d^{p-q-r} - b^{p-q-r})(a^r - b^r)$ vanishes and therefore the sum is zero. This proves that the equality holds.

IMPORTANT TERMS AND RESULTS IN ALGEBRA

1. Identities :

(a) If $a + b + c = 0, a^2 + b^2 + c^2 = -2(bc + ca + ab)$

(b) If $a + b + c = 0, a^3 + b^3 + c^3 = 3abc$

(c) If $a + b + c = 0, a^4 + b^4 + c^4 = 2(b^2c^2 + c^2a^2 + a^2b^2)$

$$= \frac{1}{2}(a^2 + b^2 + c^2)^2$$

2. Periodic function : A function f is said to be periodic, with period k..if.

$$f(x + k) = f(x) \; \forall \; x$$

3. **Pigeon Hole Principle (PHP)** : If more than n objects are distributed in 'n' boxes, then at least one box has more than one object in it.

4. **Polynomials** :

(a) A function f defined by

$$f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$$

where $a_0 \neq 0, n$ is a positive integer or zero and $a_i (i = 0, 1, 2, ..., n)$ are fixed complex numbers, is called a polynomial of degree n in x. The numbers $a_0, a_1, a_2, ..., a_n$ are called the coefficients of f. If α be a complex number such that $f(\alpha) = 0$, then α is said to be a zero of the polynomial f.

(b) If a polynomial $f(x)$ is divided by $x - h$, where h is any complex number, the remainder is equal to $f(h)$.

(c) If h is a zero of a polynomial $f(x)$, then $(x - h)$ is a factor of $f(x)$ and conversely.

(d) Every polynomial equation of degree $n \geq 1$ has exactly n roots.

(e) If a polynomial equation wish real coefficients has a complex root $p + iq$ (p, q real numbers, $q \neq 0$) then it also has a complex root $p - iq$.

(f) If a polynomial equation with **rational** coefficients has an irrational root $p + \sqrt{q}$ (p, q rational, $q > 0$, q not the square of a rational number), then it also has an irrational root $p - \sqrt{q}$.

(g) If the rational number $\frac{p}{q}$ (a fraction in its lowest terms so that p, q are integers, prime to each other, $q \neq 0$) is a root of the equation

$$a_0x^n + a_1x^{n-1} + ... + a_n = 0$$

where $a_0, a_1, ..., a_n$ are integers and $a_n \neq 0$, then p is a **divisor** of a_n and q, is a divisor of a_0.

(h) A number α is a **common root** of the polynomial equations $f(x) = 0$ and $g(x) = 0$ iff it is a root of $h(x) = 0$, where $h(x)$ is the G.C.D. of $f(x)$ and $g(x)$.

(i) A number α is a repeated root of a polynomial equation $f(x) = 0$ iff it is a common root of $f(x) = 0$ and $f'(x) = 0$.

5. **Functional equation** : An equation involving an unknown function is called a functional equation.

6. (a) If α, β be the roots of the equation $ax^2 + bx + c = 0$ then $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(b) If α, β, γ be the roots of the equation $ax^3 + bx^2 + cx + d = 0$ then,

$$\alpha + \beta + \gamma = \frac{-b}{a}; \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}; \alpha, \beta\gamma = \frac{-d}{a}$$

(c) If $\alpha, \beta, \gamma, \delta$ be the roots of the equations $ax^4 + bx^3 + cx^2 + dx + e = 0$ then,

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a}; \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma\delta = \frac{e}{a}$$

Question 1. The product of two roots of the equation $4x^4 - 24x^3 + 31x^2 + 6x - 8 = 0$ is 1, find all the roots.

Solution : Suppose the roots are $\alpha, \beta, \gamma, \delta$ and $\alpha\beta = 1$.

Now, $$\sigma_1 = (\alpha + \beta) + (\gamma + \delta) = -\frac{-24}{4} = 6 \quad ...(1)$$

$$\sigma_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{31}{4}$$

$\Rightarrow$ $$(\alpha + \beta)(\gamma + \delta) + \gamma\delta = \frac{31}{4} - 1 = \frac{27}{4} \qquad \ldots(2)$$

$$\sigma_3 = \gamma\delta(ga + \beta) + \alpha\beta(\gamma + \delta) = \frac{-3}{2}$$

$\Rightarrow$ $$\gamma\delta(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2} \qquad \ldots(3)$$

$\Rightarrow$ $$\sigma_4 = \alpha\beta\gamma\delta = -2$$

$\Rightarrow$ $$\gamma\delta = -2 \qquad \ldots(4)$$

From Eq. (2) and Eq. (4), we get

$$(\alpha + \beta)(\gamma + \delta) = \frac{35}{4} \qquad \ldots(5)$$

From Eq. (3) and Eq. (4), we get

$$-2(\alpha + \beta) + (\gamma + \delta) = \frac{-3}{2} \qquad \ldots(6)$$

From Eq. (1) and Eq. (6), we get

$$3(\alpha + \beta) = \frac{15}{2}$$

or $$\alpha + \beta = \frac{5}{2}$$

Question 2. If α, β, γ are the roots of $x^3 + px + q = 0$, then prove that

(i) $$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{\alpha^3 + \beta^3 + \gamma^3}{3} \times \frac{\alpha^2 + \beta^2 + \gamma^2}{2}$$

(ii) $$\frac{\alpha^7 + \beta^7 + \gamma^7}{7} = \frac{\alpha^5 + \beta^5 + \gamma^5}{5} \times \frac{\alpha^2 + \beta^2 + \gamma^2}{2}$$

Solution : (i) Since α, β, γ are the roots of

$$x^3 + px + q = 0, \qquad \ldots(1)$$

we have,

$$\left.\begin{aligned}\alpha^3 + p\alpha + q &= 0\\ \beta^3 + p\beta + q &= 0\\ \gamma^3 + p\gamma + q &= 0\end{aligned}\right\} \qquad \ldots(2)$$

From (2),

$$\Sigma\alpha^3 + p(\Sigma\alpha) + 3q = 0$$

But $\Sigma\alpha = 0$, from Eq. (1)

$\therefore$

$$\Sigma\alpha^3 = -3q$$

$$\begin{aligned}\Sigma\alpha^2 &= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta\\ &= 0^2 - 2 \times p \qquad (\because\ \Sigma\alpha\beta = p)\\ &= -2p \qquad \ldots(4)\end{aligned}$$

Multiplying (1) by x^2, we get

$$x^5 + px^3 + qx^2 = 0 \qquad \ldots(5)$$

and α, β, γ are three roots of Eq. (5). So

$$\left.\begin{aligned}\alpha^5 + p\alpha^3 + q\alpha^2 &= 0\\ \beta^5 + p\beta^3 + q\beta^2 &= 0\\ \gamma^5 + p\gamma^3 + q\gamma^2 &= 0\end{aligned}\right\} \qquad \ldots(6)$$

From Eq. (6), $\Sigma\alpha^5 + p\Sigma\alpha^3 + q\Sigma\alpha^2 = 0$

$$\begin{aligned}\Sigma\alpha^5 &= -(p\Sigma\alpha^3 + q\Sigma\alpha^2)\\ &= -[p(-3q) + q(-2p)] \qquad \ldots(7)\\ &= 3pq + 2pq = 5pq\end{aligned}$$

or

$$\begin{aligned}\frac{1}{5}\Sigma\alpha^5 &= pq\\ &= \left(-\frac{1}{2} \times \Sigma\alpha^2\right)\left(-\frac{1}{3}\Sigma\alpha^3\right)\\ &= \left[\frac{1}{3}\Sigma\alpha^3\right]\left[\frac{1}{2}\Sigma\alpha^2\right]\end{aligned}$$

$$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{\alpha^3 + \beta^3 + \gamma^3}{3} \times \frac{\alpha^2 + \beta^2 + \gamma^2}{2} \qquad \ldots(8)$$

Multiplying Eq. (1) by x, we get

$$x^4 + px^2 + qx = 0 \qquad \ldots(9)$$

and hence $\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$

$$\Rightarrow \qquad \Sigma\alpha^4 = -p\Sigma\alpha^2 \qquad (\because \ \Sigma\alpha = 0)$$

Again multiplying Eq. (1) by x^4, we get

$$x^7 + px^5 + qx^4 = 0 \qquad \ldots(10)$$

and hence $\Sigma\alpha^7 + p\Sigma\alpha^5 + q\Sigma ga^4 = 0$

$$\begin{aligned} \text{or} \qquad \Sigma\alpha^7 &= -p\Sigma\alpha^5 - q\Sigma\alpha^4 \\ &= -p \times 5pq - q(-p\Sigma\alpha^2) \\ &= -5p^2q - 2p^2q \\ &= -7p^2q \end{aligned}$$

$$\begin{aligned} \text{or} \qquad \frac{1}{7}\Sigma\alpha^7 &= -p^2q \\ &= pq \times (-p) \\ &= \left(\frac{1}{5}\Sigma\alpha^5\right) \times \left(\frac{1}{2}\Sigma\alpha^2\right) \end{aligned}$$

$$\text{or} \qquad \left(\frac{\alpha^7 + \beta^7 + \gamma^7}{7}\right) = \left(\frac{\alpha^5 + \beta^5 + \gamma^5}{5}\right) \times \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2}\right)$$

Question 3. Find the common roots of

$x^4 + 5x^3 - 22x^2 - 50x + 132 = 0$ and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$

hence solve the equations.

Solution : You can see that $4(x^2 - 5x + 6)$ is H.C.F. of the two equations and hence, the common roots are the roots of

$$x^2 - 5x + 6 = 0 \; i.e., \; x = 3 \text{ or } x = 2$$

Now, $$x^4 + 5x^3 - 22x^2 - 50x + 132 = 0 \qquad \ldots(1)$$

and $$x^4 + x^3 - 20x^2 + 16x + 24 = 0 \qquad \ldots(2)$$

have 2 and 3 as their common roots.

If the other roots of Eq. (1) are α and β, then $\alpha + \beta + 5 = -5$,

$$\Rightarrow \qquad \alpha + \beta = -10 \text{ from eq. (1)}$$

$$6\alpha\beta = 132$$

$$\Rightarrow \qquad \alpha\beta = 22$$

So, α and β are also roots of the quadratic equation

$$x^2 + 10x + 22 = 0$$

$$\therefore \qquad x = \frac{-10 \pm \sqrt{100 - 88}}{2} = \frac{-10 \pm 2\sqrt{3}}{2} = -5 \pm \sqrt{3}$$

So the roots of Eq. (1) are $2, 3, -5 + \sqrt{3}, -5\sqrt{3}$.

For Eq. (2), if α_1 and β_1 be the roots of Eq. 92), then we have

$$\alpha_1 + \beta_1 + 5 = -1$$

$$\alpha_1 + \beta_1 = -6$$

$$6\alpha_1\beta_1 = 24 \text{ or } \alpha_1\beta_1 = 4$$

So α_1 and β_1 are the roots of

$$x^2 + 6x + 4 = 0$$

$$x = \frac{-6 \pm \sqrt{36 - 16}}{2} = -3 \pm \sqrt{5}$$

So the roots of Eq. (2) are $2, 3, -3 + \sqrt{5}, -3 - \sqrt{5}$.

Question 4. Solve the system :

$$(x + y)(x + y + z) = 18$$

$$(y + z)(x + y + z) = 30$$

$$(z + x)(x + y + z) = 2L$$

in terms of L.

Solution : Adding the three equations, we get

$$2(x + y + z)^2 = 48 + 2L$$

or

$$x + y + z = \sqrt{24 + L}$$

Dividing the three equations by $(x + y + z) = \sqrt{24 + L}$, we get

$$x + y = \frac{18}{\sqrt{24 + L}}, \; y + z = \frac{30}{\sqrt{24 + L}}, \; z + x = \frac{24}{\sqrt{24 + L}}$$

and solving we get,

$$x = \frac{\left(\sqrt{24 + L}\right)^2 - 30}{\sqrt{24 + L}} = \frac{L - 6}{\sqrt{24 + L}},$$

$$y = \frac{(24 + L) - 2L}{\sqrt{24 + L}} = \frac{24 - L}{\sqrt{24 + L}},$$

and

$$z = \frac{24 + L - 18}{\sqrt{24 + L}} = \frac{L + 6}{\sqrt{24 + L}}.$$

Question 5. If x_1 and x_2 are non zero roots of the equation $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$ respectively, prove that $\frac{a}{2}x^2 + bx + c = 0$ has a root between x_1 and x_2.

Solution : x_1 and x_2 are roots of

$$ax^2 + bx + c = 0 \quad \ldots(1)$$

and

$$-ax^2 + bx + c = 0 \quad \ldots(2)$$

respectively.

We have

$$ax_1^2 + bx_1 + c = 0$$

and

$$-ax_2^2 + bx_2 + c = 0$$

Let

$$f(x) = \frac{a}{2}x^2 + bx + c.$$

Thus,

$$f(x_1) = \frac{a}{2}x_1^2 + bx_1 + c \quad \ldots(3)$$

$$f(x_2) = \frac{a}{2}x_2^2 + bx_2 + c \quad \ldots(4)$$

Adding $\frac{1}{2}ax_1^2$ in Eq. (3), we get

$$f(x_1) + \frac{1}{2}ax_1^2 = ax_1^2 + bx_1 + c = 0$$

$$\Rightarrow \quad f(x_1) = -\frac{1}{2}ax_1^2 \quad \ldots(5)$$

Subtracting $\frac{3}{2}ax_2^2$ from Eq. (4), we get

$$f(x_2) - \frac{3}{2}ax_2^2 = -ax_2^2 + bx_2 + c = 0$$

$$\Rightarrow \quad f(x_2) = \frac{3}{2}ax_2^2.$$

Thus $f(x_1)$ and $f(x_2)$ have opposite signs and, hence, $f(x)$ must have a root between x_1 and x_2.

Question 6. Find all real values of m such that both roots of the equation $x^2 - 2mx + (m^2 - 1) = 0$ are greater than -2 but less than $+4$.

Solution : The roots are $m \pm 1$ *i.e.*, $(m + 1), (m - 1)$

$\therefore \quad -2 < (m - 1) < (m + 1) < 4$ gives

$-1 < m < 3.$

Question 7. The roots of the equation $x^5 - 40x^4 + px^3 + qx^2 + rx + s = 0$ are in G.P. The sum of their reciprocal is 10. Compute the numerical value of $|s|$.

Solution : Let the roots be $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

$\therefore$ Sum of the root $= a\left(\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2\right) = 40$ ...(1)

Sum of be reciprocals $= \frac{1}{a}\left(r^2 + r + 1 + \frac{1}{r} + \frac{1}{r^2}\right) = 10$ ...(2)

Dividing (1) by (2), $a^2 = 4 \therefore a = \pm 2$ ...(3)

Since s is the –ve of the product of the roots $s = -a^5$ ...(4)

$\therefore$ $s = \pm 32$ or $|s| = 32$ ...(5)

Question 8. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ where a, b, c, d are constants. If

$$P(1) = 10, P(2) = 20, P(3) = 30$$

compute $\frac{P(12) + P(-8)}{10}$.

Solution : We use a trick $Q(x) = p(x) - 10x$ ...(1)

The $Q(1) = Q(2) = Q(3) = 0$ ...(2)

$\therefore$ $Q(x)$ *i.e.*, divisible by $(x-1)(x-2)(x-3)$ ...(3)

Since $Q(x)$ is a 4th degree polynomial

$Q(x) = (x-1)(x-2)(x-3)(x-r)$

and $P(x) = (x-1)(x-2)(x-3)(x-r) + 10x$ ...(4)

$\therefore$ $\frac{P(12) + P(-8)}{10} = 1984$

Question 9. Let $P(x) = 0$ be the polynomial equation of least possible degree with rational coefficients, having $\sqrt[3]{7} + \sqrt[3]{49}$ as a root, Compute the product of all the roots of $P(x) = 0$.

Solution : Let $x = \sqrt[3]{7} + \sqrt[3]{49}$

$\therefore$ $x^3 = 7 + 49 + 3 \cdot \sqrt[3]{7} \cdot \sqrt[3]{49}$

i.e., $x^3 = 56 + 21x$

Thus, $P(x) = x^3 - 21 - 56 = 0$ and the product of the root is 56.

Question 10. The equations $x^3 + 5x^2 + px + q = 0$ and $x^3 + 7x^2 + px + r = 0$ have two roots in common. If the third root of each equation is represented by x_1 and x_2 respectively, compute the ordered pair (x_1, x_2).

Solution : Common roots must be the roots of $2x^2 + (r - q) = 0$ (Difference of equation)

$\therefore$ Their sum is 0.

Then the third root of the first equation must be -5 and of the second equation is –7.

$\therefore \quad (x_1, x_2) = (-5, -7).$

Question 11. If a, b, c, x, y, z are all real and $a^2 + b^2 + c^2 = 25$, $x^2 + y^2 + z^2 = 36$ and $ax + by + cz = 30$, find the value of $\frac{a + b + c}{x + y + z}$.

Solution : $$\left(\frac{a}{5}\right)^2 + \left(\frac{b}{5}\right)^2 + \left(\frac{c}{5}\right)^2 - 2\left(\frac{ax}{30} + \frac{by}{30} + \frac{cz}{30}\right) + \left(\frac{x}{6}\right)^2 + \left(\frac{z}{6}\right)^2 + \left(\frac{z}{6}\right)^2 = 1 - 2 + 1 = 0$$

$$\therefore \left(\frac{a}{5} - \frac{x}{6}\right)^2 + \left(\frac{b}{5} - \frac{y}{6}\right)^2 + \left(\frac{c}{5} - \frac{z}{6}\right)^2 = 0$$

Thus $\frac{a}{5} = \frac{x}{6}$

$\therefore \quad a = kx$

where $k = \frac{5}{6}; b = ky$ and $c = kz$.

$$\therefore \quad \frac{a + b + c}{x + y + z} = \frac{k(x + y + z)}{x + y + z} = k$$

$$k = \frac{5}{6}$$

Question 12. If the integer A its reduced by the sum of its digits, the result is B. If B is increased by the sum of its digits, the result is A. Compute the largest 3-digt number A with this property.

Solution : A – (sum of the digits) must be divisible by 9. Then B + (sum of the digits) does not satisfy must be divisible by 9.

Now consider 999 : 999 – 27 = 972 (so defined sum of 27)

990 : 990 – 18 = 972 (so defined sum of 18)

$\therefore$ Answer is 990.

Question 13. The roots of $x^4 - kx^3 + kx^2 + lx + m = 0)$ are a, b, c, d. If k, l, m are real numbers, compute the minimum value of the sum $a^2 + b^2 + c^2 + d^2$.

Solution : Sum of the roots = k; Sum of the roots taken two at a line = $-k$

Then $k^2 = (a + b + c + d)^2 = (a^2 + b^2 + c^2 + d^2) + 2(ab + ac + ad + bc + bd + cd)$

$$= (a^2 + b^2 + c^2 + d^2) + 2k$$

Thus $a^2 + b^2 + c^2 + d^2 = k^2 - 2k$ …(1)

Thus minimum value of $k^2 - 2k = 1$.

Question 14. If $2\left[\frac{x}{6}\right]^2 + 3\left[\frac{x}{6}\right] = 20$, then it must be true that $a \le x < b$ for some integers a and b. Compute (a, b) where (b – a) as small as possible. **Note** : [x] represents the greatest integer function.

Solution : Replacing $\left[\frac{x}{6}\right]$ by y and solving, $2y^2 + 3y - 20 = 0$

$$\Rightarrow \qquad y = \frac{5}{2} \text{ or } -4$$

$$\therefore \qquad -4 \le \frac{x}{6} < -3$$

which means $-24 \le x < -18$

$\therefore$ **Ans.** (–24, 18)

Question 15. The roots of $x^3 + px^2 + qx - 19 = 0$ are each one more than the roots of $x^3 - Ax^2 + Bx - C = 0$. If A, B, C, P, Q are constants, compute $A + B + C$.

Solution : Now $(a+1)(b+1)(c+1) = 19$.

Then
$$\begin{aligned} A + B + C &= (a + b + c) + (ab + bc + ca) + (abc) \\ &= (a+1)(b+1)(c+1) - 1 \\ &= 19 - 1 \\ &= 18 \end{aligned}$$

Question 16. Find all ordered pairs of positive integers (x, z) that $x^2 = z^2 + 120$.

Solution : $x^2 - z^2 = 120$

$\Rightarrow (x + z)(x - z) = 120 = 1.120 = 2.60 = 3.40 = 4.30 = 5.24 = 6.20 = 8.15 = 10.12$

$\therefore$ $x = 31; z = 29; x = 17, z = 13; x = 13, z = 7; x = 11, z = 1$

$\therefore$ Required ordered pairs are : (31, 29), (17, 13), (13, 7), (11, 1).