Chapter 4

Geometry

1. The internal (or external) bisector of $\angle A$ of ΔABC divides the opposite side *BC* internally (or externally) in the ratio of the sides *AB* and *AC* containing the $\angle A$.

   Definition : if a line segment *AB* is divided internally and externally in the same ratio at *P* and *Q* respectively, then *AB* is said to be divided harmonically at *P* and *Q*. The points *P* and *Q* are called harmonic conjugates with respect to *AB*.

   Note :

   (a) If *P* and *Q* are harmonic conjugates with respect to *AB* hen *A* and *B* are harmonic conjugates with respect to *PQ*.

   (b) If *P* and *Q* divide *AB* harmonically, then *AP*, *AB* and *AQ* are in harmonic progression, *i.e.*,

$$\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}.$$

   (c) If *O* is the mid point of *AB* and *P*,*Q* divide *AB* harmonically then $OB^2 = OP.OQ$.

2. Given a harmonic range (*ABCD*) and a point *O* outside the line *ABCD*, if a parallel through *B* to *OA* meets *OC*, *OD* at *P*,*Q* and $PB = BQ$ then *ABCD* is a harmonic range.

3. Given four collinear points *A*, *B*, *C*, *D* and a point outside that line, if the parallel through *B* to *OA* meets *OC*, *OD* at *P*,*Q* and $PB = BQ$ then *ABCD* is a harmonic range.

4. Given a harmonic range *ABCD* and a point *O* out side the line *ABCD*, any transversal cuts the four lines *OA*, *OB*, *OC*, *OD* in four harmonic points.

   Definition : Four concurrent lines *OA*, *OB*, *OC*, *OD* which are cut by one transversal, and therefore by every transversal, in four harmonic points are said to form a harmonic pencil.

5. If two conjugate rays of a harmonic pencil are rectangular, they are the bisectors of the angles formed by the other to rays of the pencil.

   Item if *C*, *D* divide *AB* harmonically, and *C'*, *D'* divide *A'B'* harmonically and if *AA'*, *BB'*, *C'* meet in a point *O* then *DD'* passes through *O*.

6. If *ABCD* and *AB'C'D'* are tow harmonic ranges and the lines *ABCD* and *AB'C'D'* are distinct, then the lines *BB'*, *CC'*, *DD'* are concurrent.

7. The areas of two similar triangles are proportional to the squares on corresponding sides.

8. In a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the sides containing the right angle. (*Pythagoras Theorem*)

9. If A, B, C are three collinear points and P any other point, then

$$PA^2.BC + PB^2.CA + PC^2.AB = 0.$$

$$PA^2.BC + PB^2.CA + PC^2.AB = 0$$

(using directed line segments) (Stewart's Theorem).

10. (Menelaus's Theorem) if a transversal cuts the sides BC, CA, AB of a triangle ABC at D, E, F respectively then

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1.$$

11. (Converse of Menelaus's Theorem) If D, E, F are points on the sides BC, CA, AB of a ΔABC such that

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$$

then D, E, F are collinear.

12. (Ceva's Theorem) If the lines joining the vertices A, B, C of ΔABC to any points S in their plane meet the opposite sides in D, E, F respectively, then

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1.$$

13. (Converse of Ceva's Theorem) If three Cevians AD, BE, CF satisfy

$$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1$$

then they are concurrent. (By a Cevian we mean a line segment joining a vertex of a triangle t any given point on the opposite side).

14. A circle is symmetrical about any of its diameters.

15. Given any three non-collinear points A, B, C there exists a unique circle passing through A, B, C.

16. Equal chords of a circle are equidistant from the center.

17. Given any two chords of a circle, the one which is nearer to the center is greater than the one more remote.

18. For any are of a circle, the angle subtended at the center is double the angle subtended at any point on the remaining part of the circumference.

19. Angles in the same segment of a circle are equal.

20. If a straight line segment joining two points subtends equal angles at two other points on the same side of it, then the four points are concyclic.

21. One and only one tangent can be drawn to a circle at any point on its circumference and this tangent is perpendicular to the radius through the point of contact.

22. If two tangents are drawn to a circle from an exterior point then

    (i) the lengths of the tangents are equal

(ii) they subtend equal angles at the center

(iii) the angle between them is bisected by the straight line joining the point and the center.

23. If two circles touch one another, then the point of contact lies on the straight line joining the centers.

24. In equal circles (or in the same circle) if two chords are equal, then they cut off equal arcs on the circles.

25. In any circle the angle between a tangent and a chord through the point of contact of the tangent is equal to the angle in the alternate segment.

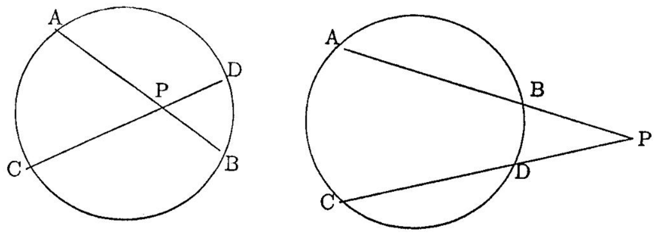
26. A common tangent to two circles divides the straight line segment joining the centers, externally or internally in the ratio of their radii. (The points S and S' dividing the line segment joining the centers of two circles in the ratio of their radii are known as the centers of similitude of the two circles. The two common tangents from the external center of similitude are the direct common tangents and the two common tangents and the two common tangents from the internal center of similitude are the transverse common tangents.

27. The opposite angles of a cyclic quadrilateral are supplementary (*i.e.*, they add up to 180º).

28. If $ABCD$ is a cyclic quadrilateral then any exterior angle of $ABCD$ is equal to the interior opposite angle.

29. If two opposite angles of a quadrilateral are supplementary then it is cyclic.

30. If AB and CD are any two chords of circle meeting at a point P then $PA.PB = PC.PD$ (known as the secant property of a circle).



31. If P is any point on a chord AB (or AB produced) of a circle with center O and radius r, then

$$AP.PB = r^2 - OP^2 \text{ or } PA.PB = OP^2 - r^2$$

according as P is within the circle or outside the circle.

32. If P is any point on a chord AB produced of a circle with center O and radius r then

$$PA.PB = PT^2 = (\text{length of the tangent from } P)^2$$

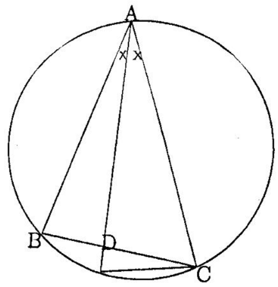
Definition : If P is any point in the plane o a circle with center O and radius r, the power of P with respect to the circle is defined as $OP^2 - r^2$. Thus if directed segments are used then $PA.PB =$ Power of P with respect to the circle whenever P is a point on the chord AB (or AB produced).

Note : If p lies on the circle Σ, then the power of P with respect to the circle Σ is zero; if P lies outside the circle, then the power of P is the square of the length of the tangent from P, and if P lies inside the circle, the power of P is negative.

33. If two straight line segments AB and CD (or both being produced) intersect at P so that $PA.PB = PC.PD$ then the four points A, B, C and D are concyclic.

34. If AD bisects the vertical angle A of ΔABC meeting the base BC at D then

$$AB.AC = BD.DC + AD^2$$



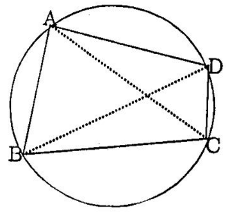
35. If AD is the altitude through A of ΔABC and if R is the circum radius of ΔABC then $AB.AC = 2R.AD.$

36. $\Delta =$ Area of $\Delta ABC = \dfrac{abc}{4R}.$

37. (Ptolemy's Theorem) The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by pairs of opposite sides.

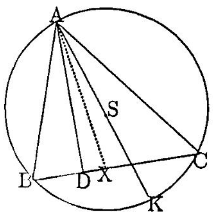
38. If $ABCD$ is a quadrilateral which is not cyclic, then

$$AB.CD + BC.AD > AC.BD.$$

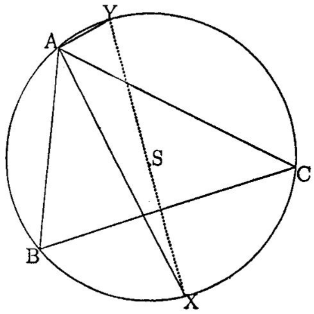


39. Quadrilateral $ABCD$ is cyclic if and only if $AC.BD = AB.CD + AD.BC.$

40. In ΔABC let AD be the altitude through A and AK the circum diameter through A. Then $\angle DAK = \angle B - \angle C.$ Further the angular bisector AX of $\angle A$ bisects $\angle DAK.$



41. In $\Delta ABC,$ if the internal and external bisectors of $\angle A$ meet of the circum circle at X and Y, then XY circum diameter perpendicular to BC.



42. A triangle and its medial triangle have the same centroid.

43. If m_a, m_b, m_c are the lengths of the medians of $\Delta ABC,$ through A, B, C respectively then

$$2m_a^2 = b^2 + c^2 - \frac{a^2}{2},$$

$$2m_b^2 = c^2 + a^2 - \frac{b^2}{2},$$

$$2m_c^2 = a^2 + b^2 - \frac{c^2}{2}.$$

where a, b, c are the lengths of the sides BC, CA, AB of $\Delta ABC.$

44. If G is the centoid of $\Delta ABC,$ then

1. $1.m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

2. $GA^2 + GB^2 + GC^2 = \frac{1}{3}(a^2 + b^2 + c^2)$

45. If P is any point in the plane of ΔABC and G is the centroid of ΔABC then

$$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3PG^2.$$

46. If R is the circumradius and S is the circumcenter of ΔABC then

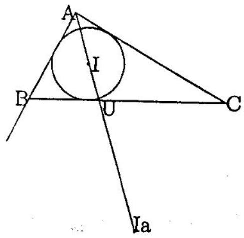
$$SG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2).$$

47. In any $\Delta ABC,$ if $\angle B > \angle C$ then the internal bisector BE of $\angle B$ is shorter than the internal bisector CF of $\angle C.$

48. If two internal bisectors are equal then the triangle is isosceles.

49. The external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle.

50. The incenter I and the excenter I_a opposite to A divide the bisector AU harmonically, where U is the point of intersection of the internal bisector of $\angle A$ and $BC.$



51. If is the incenter and I_a is the excenter opposite to A then

$$AI.AI_a = AB.AC.$$

52. If the incircle of ΔABC touches the sides BC, CA, AB of the triangle at X, Y, Z then

$$BX = s - b,\ CY = s - c \text{ and } AZ = s - a \text{ where } 2s = a + b + c.$$

53. If the escribed circle opposite to A touches the sides BC, CA, AB of ΔABC at X_a, Y_a, Z_a respectively, then

$$AZ_a = AY_a = s = BX_b = CY_c = CX_c.$$

(with obvious meaning for $X_b, Y_b, Z_b, X_c, Y_c, Z_c$)

54. (notations as in the previous problem)

$$BX_a = BZ_a = s - c;\ CX_a = CY_a = s - b;\ AY_a = AZ_a = s - c.$$

Definition : Two points on a side of a triangle are isotomic points if they are equidistant from the midpoint of this side.

55. The points of contact of a side of a triangle with the incircle and excircle corresponding the this side are two isotomic points.

46. (a) $XX_a = b - c;\ YY_b = a - c;\ ZZ_c = a - b$

(b) $ZZ_a = YY_a = a;\ ZZ_b = XX_b = b;\ YY_c = XX_c = c$

57. (a) X_c and X_b are isotomic points on BC.

Y_a and Y_c are isotomic points on CA;

Z_a and Z_b are isotomic points on AB.

Further $X_bX_c = b + c,\ Y_cY_a = c + a$ and

(b) $Y_bY_c = Z_bZ_c = a.$

58. (a) For any triangle the area is equal to the product of the in radius and the semiperimeter. *i.e.*, for any ΔABC, rea of $\Delta ABC = \Delta = rs$.

(b) $\Delta = r_a(s - a) = r_b(s - b) = r_c(s - c).$

59. $rr_a = (s - b)(s - c).$

60. Area of $\Delta ABC = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (Hero's Formula).

61. (a) $rr_ar_br_c = \Delta^2.$

(b) $\frac{1}{r_1} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}.$

(c) For any ΔABC if the incircle touches the sides BC, CA AB at X, Y, Z respectively, then

$$\frac{\text{area of } \Delta ABC}{\text{area of } \Delta XYZ} = \frac{2R}{r}$$

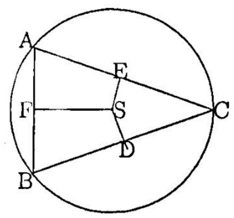
which R is the circumradius and r is the inradius of ΔABC.

52. (a) $\frac{1}{r} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$

(b) $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}.$

63. $r_a + r_b + r_c = r + 4R$

64. $SD + SE + SF = R + r.$



65. (Euler's Theorem) If d is the distance between the dicrumcenter and the incenter of a triangle then

$$SI^2 = d^2 = R^2 - 2Rr.$$

66. $(SI_a)^2 = R^2 + 2Rr_a;\ (SI_b)^2 = R^2 + 2Rr_b;\ (SI_c)^2 = R^2 + 2Rr_c.'$

67. $(II_a)^2 = 4R(r_a - r);\ (I_bI_c)^2 = 4R(r_br_c)$

68. f the line of centers of the two circles (S, R) and (I, r) satisfy

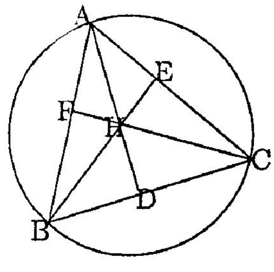
$$SI^2 = R^2 - 2Rr$$

hen an infinte number of triangles may be found such that (I, r) is the incircle of each one of them and (S, R) is the circumcircle of each one of them.

69. et ABC be a triangle with AD, BE, CF as the altitudes and H the orthocenter. Then

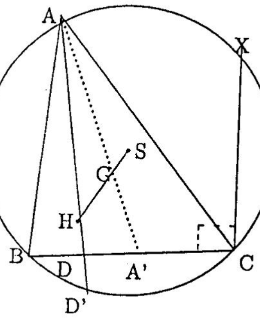
(a) $AH.HD = BH.HE = CH.HF.$

(b) $AH.HD = \frac{1}{2}(a^2 + b^2 + c^2) - 4R^2.$



70. $HA^2 = 4R^2 - a^2; HB^2 = 4R^2 - b^2; HC^2 = 4R^2 - c^2.$

71. The chord CX of the circumcircle of ΔABC perpendicular to BC is equal to AH, where H is the orthocenter of ΔABC.



72. $AH = 2SA'$ (see the figure above).

73. In any triangle the circumcenter, the orthocenter and the centroid are collinear. The centroid trisects the line joining the circumcenter and the orthocenter. (The line is known as the *Euler line* of the triangle).

74. (a) $SH^2 = 9R^2 - (a^2 + b^2 + c^2)$

(b) $GH^2 = 4R^2 - \frac{4}{9}(a^2 + b^2 + c^2).$

(c) $HA^2 + HB^2 + HC^2 = 12R^2 - (a^2 + b^2 + c^2).$

75. If the altitude AD of ΔABC meets the circumcircle again at D', then D si the midpoint of hD' where H is the orthocenter of $\Delta ABDC$. The line segment between the orthocenter and the other point of intersection with the circumcircle is bisected by the corresponding side of the triangle. (See the figure in Problem 66).

76. (see the figure in Problem 66).

77. The circumcircle of ΔHBC and the circumcircle of ΔABC have the same radius.

Definition : If D, E, F are the feet of the altitudes of a ΔABC on the corresponding sides then ΔDEF is the *Pedal triangle* or the *orthic triangle* of ΔABC.

78. The three triangles cut off from a given triangle by the sides of its Pedal triangle and the given triangle itself are mutually similar.

79. A is the midpoint of the arc $F'E'$ of the circumcircle of ΔABC; B is the midpoint of the arc $F'D'$ and C is the midpoint of the arc $D'E'$ where D', E', F' are the points where the altitudes AD, BE, CF meet the circumcircle.

80. The radii of the circumcircle through the vertices of a triangle are perpendicular to the corresponding sides of the Pedal triangle.

81. The orthocenter of an acute angled triangle is the incenter of the Pedal triangle.

82. The sides of a triangle bisect externally the angles of the Pedal triangle.

83. $AH + r_a = BH + r_b = CH + r_c = 2R + r$. (Usual notations).

84. If DEF is the Pedal triangle of an acute angled ΔABC then

$$\frac{EF}{BC} + \frac{FD}{CA} = \frac{DE}{AB} = \frac{R + r}{r}.$$

85. Perimeter of the Pedal triangle of ΔABC is $\dfrac{2 \text{ area of } \Delta ABC}{R}$.

86. ***The Nine-Point Circle Theorem*** : The feet of the three altitudes of any triangle, the midpoints of the three sides and the midpoints of the segments from the orthocenter to the three vertices all lie on a circle of radius equal to half the circumradius. Also, the center of this circle bisects the line joining the orthocenter and the circumcenter.

87. The sum of the powers of the vertices of a triangle ABC with respect to its none-point circle is $\frac{1}{4}(a^2 + b^2 = c^2)$.

88. All triangles inscribed in a given circle and having a given point as the othocenter have the same nine-point circle.

89. ***Feuerbach's Theorem*** **:** In any triangle, the nine-point circle touches the incircle and the three escribed circle.

90. ***Pedal Line Theorem*** **:** The feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies o the circumcircle.

    Definition : *If* P_1, P_2, P_3 are the feet of the perpendiculars from a point P onto the sides f ΔABC, then $\Delta P_1P_2P_3$ is called the Pedal triangle of P with respect to ΔABC.

    If p lies on the circumcircle then the Pedal Triangle of P gets degenerated into a straight line, known as the *Simson Line* of P.

91. The sides of the Pedal triangle of P with respect to ΔABC are given by

$$P_2P_3 = a\left(\frac{AP}{2R}\right); \; P_3P_1 = b\left(\frac{BP}{2R}\right); \; P_1P_2 = c\left(\frac{CP}{2R}\right),$$

    where R is the circumrdius of ΔABC.

92. If ABC is a triangle and P is not on the arc CA of the circumcircle of ΔABC, then $AB.CP + BC.AP$ $AC.BP$.

93. If $A_1B_1C_1$ is the Simson line of a point P on the circumcircle of ΔABC, then the triangles PA_1B_1 and PBA are similar. Further,

    (a) $PA.PA_1 = PB.PB_1 = PC.PC_1$

    (b) $\dfrac{PA_1.B_1C_1}{a} = \dfrac{PB_1.C_1A_1}{b} = \dfrac{PC_1.A_1B_1}{c}$

94. If P is a point on the circumcircle of ΔABC, then the Simson line of P bisects the line joining P and the orthocenter of the triangle.

95. ***Erdos-Mordell Theorem :*** If O is any point inside ΔABC and P, Q, R are the feet of the perpendiculars from O upon the respective sides BC, CA, AB of ΔABC then

$$OA + OB + OC \geq 2(OP + OQ + OR).$$

96. If no angle of ΔABC is greater than equal to 120° and equilateral triangles *AC'B*, *BA'C* and *CB'A* are constructed outwardly on the sides *AB*, *BC*, *CA* of ΔABC then the lines *AA'*, *BB*, *CC'* concur at the *Fermat point P* of $\Delta ABC,$ and further $AA' = BB' = CC'$.

98. If *C'AB*, *A'BC* and *B'CA* are the equilateral triangles drawn outwardly on the sides of a given ΔABC then the centers *X*, *Y*, *Z* of the equilateral triangles form another equilateral triangle.

99. Construct equilateral triangles on the sides of a triangle *ABC* inwardly. Then the centers *X*, , *Z* of these triangles themselves form the vertices of an equilateral triangle.

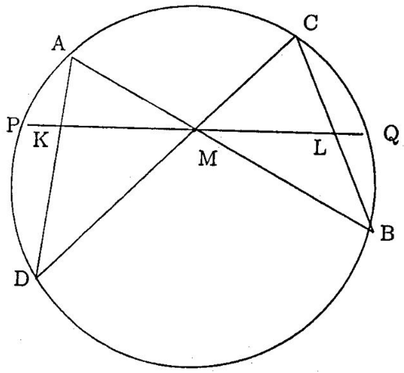
Note **:** The equilateral triangles *XYZ* are called the *Outer Napolean Triangle* and *Inner Napolean Triangle* respectively.

100. Let P_1, P_2 be any two points n the circumcircle of ΔABC. Then the angle between the Simson lines of P_1 and P_2 is half the angular measure of arc P_1P_2.

101. If P_1 and P_2 are two diametrically opposite points on the circumcircle of $\Delta ABC,$ then their Simson lines are perpendicular to each other and intersect on the nine-point circle of ΔABC.

102. Let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles inscribed in the same circle. If *P* is point on this circle, the angle between the Simson lines of *P* with respect to the two triangles is a constant.

103. ***Butterfly Theorem*** **:** *PQ* is a chord of a circle. Through the midpoint *M* of *PQ* chords *AB* and *CD* are drawn. *AD* and *BC* meet *PQ* at *K* and *L*. Then *M* is the midpoint of *KL*.



104. ***Morley's Theorem*** **:** The points of intersection of the adjacent trisectors of the angles of any triangle form the vertices of an equilateral triangle.

105. The figure formed when the midpoints of the sides of a quadrangle are joined in order is a parallelogram, and its area is half that of the quadrangle.

106. The segments joining the midpoints of pairs of opposite sides of a quadrangle and the segment joining the midpoints of the diagonals are concurrent and bisect one another.

107. If a quadrangle *ABCD* has its opposite sides *AD* and *BC* (produced) meeting at *Z* and if *X* and *Y* are the midpoints of the diagonals *AC* and *BD*, then area $\Delta XYZ = \frac{1}{4}$ area quad(*ABCD*).

108. Any four unequal lengths, each less than the sum of the other three, can serve as the sides of three different cyclic quadrangles all having the same area.

109. The area of a cyclic quadrangle is a symmetric function of its for sides.

110. ***Brahmagupta's formula*** **:** If a cyclic quadrangle has sides a, b, c, d and semiperimeter s then its area Δ is given by

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d).$$

111. ***Brahmaupta's Theorem*** **:** If a cyclic quadrangle ahs perpendicular diagonals intersecting at P, then the line through P perpendicular to any side bisects the opposite side.

112. ***Pappu's Theorem*** **:** If A, C, E are three points on one line and B, D, F on antoher, and if the three lines AB, CD, EF¸ meet DE, FA¸BC respectively, then the three points of intersection L, M, N are collinear.

Definition : If two specimens of a figure, consisting of points and lines, can be put in correspondence in such a way that pairs of corresponding points are joined by concurrent lines, the two specimens are perspective from a point. If the correspondence is such that pairs of corresponding lines meet at collinear points meet at collinear points then the two specimens are perspective from a line.

113. ***Desargues's Theorem*** **:** If two triangles are perspective from a point and if their paris of corresponding side meet, then the three points of intersection are collinear.

114. **Converse of Desargues's Theorem :** If two triangles are perspective from a line, and if two pairs of corresponding vertices are joined by intersecting lines, then the two triangles are perspective from the points of intersection of these lines.

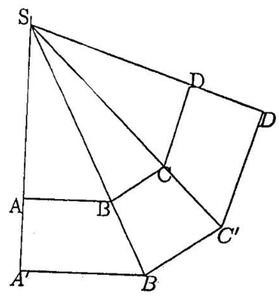
115. ***Pappu's Theorem*** **:** If each set of three alternate vertices of a hexagon is a set of three collinear points, and the three pairs of opposite sides intersect, then the three points of intersection are collinear.

116. ***Pascal's Theorem*** **:** If all the vertices of a hexagon lie on a circle and the three pairs of opposite sides intersect, then the three points of intersection are collinear.

117. ***Brianchon's Theorem*** **:** If al six sides of a hexagon touch a circle, the three diagonals are concurrent (or possibly parallel).

Definition : If the two corresponding sides of two similar polygons are parallel, hen the two polygons are said to be *homothetic.*

118. The lines joining corresponding vertices of two homothetic polygons are concurrent.

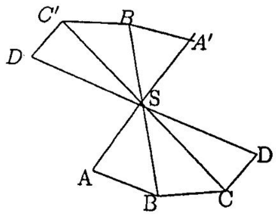


Definition : The point of concurrence (See the figure above) S is called the *homothetic center* of the two polygons.

The constant ratio $\frac{SA'}{SA} = ... = \frac{A'B'}{AB} = k$ is called the homothetic ratio.

The homothetic ratio k may be positive or negative. If k is positive the polygons are directly homothetic, if k is negative then they are inversely homothetic.

Example : This is an example with homothetic ratio equal to –1. Here the homothetic center is the mid-point of the segment determined by any two corresponding points.



Remark : We can generalize *homothetic polygons* to *homothetic figures.*

119. Given two homothetic figures, if a point on one of them describes a straight line, then the corresponding homologous point on the other figure also describes a straight line; further the two lines are parallel.

120. Corresponding angles in two homothetic figures are equal.

121. Given two homothetic figures, if a point on one of them describes a circle, then the corresponding homologous point on the other figure also describes a circle.

(a) The centers of the two-homothetic circles are corresponding points in the two homothetic figures.

(b) The ratio of the radii of the two circles is equal to the homothetic ratio.

122. A triangle ABC and its medial triangle $A'B'C'$ are homothetic with the centroid G as the homothetic center and –2 as the homothetic ratio.

123. The nine-point circle and the circumcircle of a ΔABC are homothetic with the orthocenter H as the homothetic center and 2 as the homothetic ratio; they are also homothetic with G as center and homothetic ratio –2.

124. The triangle formed by the tangents a tangential triangle to the circumcircle at the vertices of a ΔABC and the pedal triangle of ΔABC are homothetic.

125. The homothetic center of the pedal triangle and the tangential triangle lies on the Euler line of the given triangle.

126. The circumcenter of the tangential triangle of a given triangle lies on the Euler line of the given triangle.

127. If p be the inradius of the pedal triangle, R the circumradius of ΔABC and q the circumradius of the tangential then the homothetic ratio of the tangential triangle and the pedal triangle is $R : p$ of their inradii, and also is $q : \frac{R}{2}$ of their circumradii. This gives

$$R^2 = 2pq.$$

Trigonometric Formulae

128. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$

129. $a = b\cos C + c\sin C.$

130. $a^2 = b^2 + c^2 - 2bc \cos A.$

131. $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$

132. $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}.$

133. $\tan \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}.$

134. $\text{Area } = \dfrac{1}{2} bc \sin A.$

135. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$

136. $\Delta = \dfrac{abc}{4R}.$

137. $r = \dfrac{\Delta}{s} = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = s-c)\tan\dfrac{C}{2} = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}.$

138. $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

139. $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2}$

140. $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4R\sin\dfrac{C}{2}\cos\dfrac{A}{2}\cos\dfrac{B}{2}$

141. $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = \dfrac{a^2+b^2+c^2}{\Delta^2}$

142. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

143. $rr_1r_2r_3 = \Delta^2.$

144. $r_2r_3 + r_3r_1 + r_1r_2 + r_1r_2 = s^2$

145. $r_1 + r_2 + r_3 - r = 4R.$

146. $r_1r_2r_3 = rs^2.$

147. For the excentral triangle of ΔABC we have

(a) its angles are $\dfrac{\pi - A}{2}, \dfrac{\pi - B}{2}, \dfrac{\pi - C}{2};$

(b) its sides are $4R\cos\dfrac{A}{2}, 4R\cos\dfrac{B}{2}, 4R\cos\dfrac{C}{2};$

(c) its circumradius is $2R$.

(d) its in radius is $2R\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1\right)$

(e) its area is $2Rs$.

148. If X, Y, Z are the points of contact of the incircle of a ΔABC with its sides then

(a) the sides of ΔXYZ are $2r\cos\frac{A}{2}$, $2r\cos\frac{B}{2}$, $2r\cos\frac{C}{2}$;

(b) its angles are $\frac{\pi-A}{2}$, $\frac{\pi-B}{2}$, $\frac{\pi-C}{2}$;

(c) its area = $Rr\sin A\sin B\sin C = \frac{\Delta r}{2R}$

149. In a triangle ΔABC,

(a) $IA.IB.IC = 4Rr^2$

(b) $II_1.II_2.II_3 = 16R^2r$.

(c) $II_1^2 + I_2I_3^2 = II_2^2 + II_3^2 + I_1I_2^2$.

(d) $I_1A.I_1B.I_1C = 4Rs^2$.

150. For the pedal triangle

(a) its sides are $a\cos A = R\sin 2A$; $b\cos B = 2R\sin 2B$; $c\cos C = R\sin 2C$.

(b) its angles are $\pi-2A$, $\pi-2B$, $\pi-2C$.

151. For the ΔABC, we have

(a) the medians have lengths

$$m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2},\ m_b = \frac{1}{2}\sqrt{2c^2+2a^2-b^2},\ m_c = \frac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

(b) $AI = \frac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$

(c) $SI^2 = R^2\left[1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right] = R^2-2Rr$.

(d) $SI_1^2 = R^2\left[1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right] = R^2+2Rr_1$.

(e) $IH^2 = 2r^2-4R^2\cos A\cos B\cos C$

(f) $SH^2 = R^2(1-8\cos A\cos B\cos C) = 9R^2-a^2-b^2-c^2$

(g) $I_1H^2 = 2r_1^2-4R^2\cos A\cos B\cos C$.

(h) $IN = \frac{R}{2}-r$.

(i) $I_1N = \frac{R}{2} + r_1.$

(j) $SG^2 = R^2 - \frac{a^2 + b^2 + c^2}{9}.$

(k) $AH^2 + BH^2 + CH^2 = 3R^2.$

(l) $SI^2 + SI_1^2 + SI_2^2 + SI_3^2 = 12R^2.$

Definition : Given a circle Σ with center *O* and radius *k*, the inverse of a point *P* with respect to Σ with respect to Σ is the point *Q* on the ray *OP* such that

$$OP.OQ = k^2.$$

Two inverse points lie on the same side of the center of the circle.

Of two inverse points one lies inside the circle and the other lies outside the circle.

For a point on the circle, its inverse coincides with itself.

152. Two inverse points divide the corresponding diameter harmonically. Conversely, if two points divide a diameter o a circle harmonically then they form a pair of inverse points with respect to the circle.

153. The ratio of the distances of a variable point on a circle from two given inverse points is constant.

Definition : For any point *P* the triangle formed by the feet of the perpendiculars from *P* on the sides of a ΔABC is called the *pedal triangle* of *P* with respect to the given triangle *ABC*.

154. If *p* and *P'* are inverse points with respect to the circumcricle of $\Delta ABC,$ then their pedal triangles with respect to ΔABC are similar.

155. If the pedal triangles of two points for a given triangle ΔABC are similar then the two points are inverse points with respect to the circumcircle of $\Delta ABC.$

Definition : Two circles are orthogonal if the tangents to the two circles at any point of their intersection are orthogonal.

Conversely, if the two radii passing through a point common to two circles are perpendicular, then the two circles are orthogonal.

156. If two circles are orthogonal the circle having for diameter their line of centers passes through their common point.

157. If the circle on the line of centers of two given circles as diameter, passes through a common point of the two given circles, then the given two circles are orthogonal.

158. If two circles are orthogonal, the radius of one circle passing through a point common to the two circles is tangent to the second circle.

159. If two circles are orthogonal, any two points of one of them collinear with the center of the second are inverse point with respect to the second circle.

160. If two points of one circle are inverse points for a second circle, then the two circles are orthogonal.

161. If *A*, *C*, *B*, *D* form a harmonic range in that order, then the circle on *AB* as diameter is orthogonal to every circle passing through *C* and *D*.

Definition : Given two circles with centers A, B and radii r_1 and r_2 the points S and S' dividing AB externally and internally in the ratio $r_1 : r_2$ are called the external and internal centers of similitude of the given two circles.

162. The circles are homothetic in two and only two ways.

Remark :

(a) The external center of similitude is collinear with the ends of any two parallel radii directed in the same sense. The internal center of similitude is collinear with the ends of any two paralllel radii lying on the opposite sides of the line of centers.

(b) The centers of the two circles and the two centers of similitude of he two circles are two paris of harmonic points.

(c) If the circles are tangent to each other, their point of contact is a center of similitude of the two circles.

(d) Two equal circles have only one center of similitude, namely, the midpoint of their line of centers.

(e) If two circles are concentric, their common center is their only center of similitude.

163. If two circles have external common tangents, these tangents pass through the external center of similitude of the two circles.

164. If two circles have internal common tangents, these tangents pass through the internal center of similitude.

Definition : The circle having as diameter the segment determined by the two centers of similitude of two circles is called the *circle of similitude* of the two given circles.

165. If P is any point on the circle of similitude of two given circles $S(A, r_1), S(B, r_2)$ then the ratio $PA : PB = r_1 : r_2$.

166. For two intersecting circles, the circle of similitude passes through the points common to the two circles.

167. The six centers of similitude of three given circles taken in pairs lie on four straight lines.

168. A line joining two of the centers of similitude of three circles taken in pairs passes through a chord center of similitude.

169. If C is a point on the circle of similitude of the two circles $S(A, a)$ and $S(B, b)$ and if P and Q are inverses of C with respect to the two given circles $S(A, a)$ and $S(B, b)$, then P and Q are symmetrical with respect to the radical axis of the two circles.

Inversion : The inversion for which O is the center and k is the radius is denoted by (O, k^2). If the points P, P' correspond to each other in the inversion (O, k^2), and the point P describes a curve C, then the curve C' described by P' is called the inverse of the curve C.

170. Two points A, B and their corresponding points A', B' are either collinear or concyclic.

171. The inverse of a straight line not passing through the center of inversion is a circle passing through the center of inversion.

The given line is the radical axis of this circle and circle inverse of the line.

The radius of the inverse circle is equal to $\frac{k^2}{2d}$ where d is the distance of the given line from O the center of inversion.

172. The inverse of a circle passing through the center of inversion is a straight line perpendicular to the line joining the centers of inversion and the center of the given circle.

173. A straight line and a circle can be considered inverse figures in two different ways.

174. The inverse of a circle not passing through the center of inversion is again a circle not passing through the center of inversion.

Note :

(a) The center of inversion is a center of similitude of the given circle and its inverse.

(b) The two inverse points P, P' are two antihomologous points on the two inverse circles.

(c) If R and R' are the radii of the given circle and its inverse circle, then $R' = R\frac{k^2}{p}$ where p is the power of the center of inversion with respect to the given circle.

(d) When k^2 is equal to the power of the center of inversion with respect to the given circle, then the circle inverts into itself.

175. A circle orthogonal to the circle of inversion inverts into itself.

176. The center of a circle inverts into the inverse, with respect to the inverse circle, of the center of inversion.

177. Any two circles may be considered as the inverse to one another, in two different ways.

178. If two curves intersect, their angle of intersection is equal to the angle of intersection of their inverses at the corresponding point.

179. Inversion preserves orthogonality and inversion preserves tangency.

180. In an inversion two points two points inverse with respect to a circle invert into two points inverse with respect to the inverse circle of the given circle.

181. Two inverse circles and the circle of inversion are coaxial.

182. For a suitable circle of inversion, any three distinct points A, B, C can e inverted the vertices of a triangle $A'B'C'$ congruent to a given triangle.

183. Under inversion (O, k^2), for corresponding points we have

$$A'B' = \frac{k^2 AB}{OA.OB}$$

184. In an inversion (O, k^2) for any point P not on the circle of inversion, the inverse P' is he second intersection of any two circles through P orthogonal to the circle of inversion.