Chapter 2

Inequalities

1. Inequality exists only between two real numbers (not complex numbers).
2. If a be any real number then one and only one of there hold.

$$a > 0, \; a = 0, -a > 0$$

3. If $a, b > 0$ then $a + b > 0, ab > 0.$
4. (i) $a > b$ if $a - b > 0$

   (ii) $a < b$ if $b > a$

   (iii) $a \geq b$ if either $a > b$ or $a = b$

   (iv) $a \leq b$ if either $a < b$ or $a = b$
5. In a given inequality, terms/coefficients from one side to other side can be transferred as in the case of an equality.
6. On can add/subtract the same real number on both sides of an inequality, the direction of inequality does not change.
7. Two is equalities with same direction can be added (always) and multiplied (if both sides of the inequality are positive). But they can never be subtracted or divided.
8. Both sides of an inequality can be multiplied by same positive quantity without changing the direction of inequality.
9. The direction of inequality changes if it is multiplied both sides by a negative number.
10. If $a > b$ then $ac > bc$ if $a, b, c > 0.$
11. If $a < b$ then $f(a) < f(b)$, if $f(x)$ is an increasing function of x and also if $a < b,$ then $f(a) > f(b)$ if $f(x)$ is decreasing function of x.
12. For a closed convex polygon in XY-plane, any linear function of x and y *i.e.*, $z = ax + by$ defined over such a convex polygon will have maximum and minimum value only at the vertices of the polygon.
13. If $a > c$ and $b < c$ then $b < c < a$ or $b < a.$ Also if $a > b$ and $b > c$ then $a > c.$
14. Functions $f(a, b, c), g(a, b, c)$ and $h(a, b, c)$ can not be positive simultaneously if $f + g + h = 0$

15. **Jensen's Inequality** : If $f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$ for $x, y, \frac{(x+y)}{2} \in [a, b]$

then $f(x)$ is positive and curves towards x-axis at least in the given domain.

If $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$ for $x, y, \left(\frac{x+y}{2}\right) \in [a, b]$

then the function curves away from x-axis.

16. Method of induction is very useful in proving result of inequality involving only natural number.

17. If $a_1 \geq a_2, a_2 \geq a_3, ..., a_{n-1} \geq a_n$ then $a_1 \geq a_n$.

18. If $a_1 \geq b_1, a_2 \geq b_2, ..., a_n \geq b_n$ then $a_1 + a_2 + a_3 + ... + a_n \geq b_1 + b_2 + ... + b_n$ and equality sign holds iff $a_1 = b_1, a_2 = b_2, ..., a_n = b_n$.

19. (i) If $a > b > 0$ then $\frac{1}{b} > \frac{1}{a}$.

(ii) If $a > b > 0$ and $c > d > 0$ then $\frac{a}{d} > \frac{b}{c}$ and $ac > bd$.

(iii) If $a \geq b > 0$ and $a \geq d > 0$ then $\frac{a}{d} \geq \frac{b}{d}$. The equality sign holds iff $a = b$ and $c = d$.

20. (i) Let $a > b > 0, p, q > 0$ and let $a^{1/q}$ and $b^{1/q}$ denote positive q^{th} roots of a and b respectively then

$$a^{p/q} > b^{p/q} \text{ and } b^{-p/q} > a^{-p/q}$$

(ii) Let $a \geq b > 0, p, z$ e a non negative integer and q a positive integer and $a^{1/q}, b^{1/q}$ denote q^{th} roots of a and b respectively then

$$a^{p/q} \geq b^{p/q} \text{ and } b^{-p/q} \geq a^{-p/q}$$

The equality sign holds of and only if $a = b$ or $p = 0$.

21. (i) For two positive numbers a and b, let arithmetic denotes A. mean, G denotes geometric mean and 'H' denotes Halmount mean then $A \geq G \geq H$.

The equality sign holds if and only if $a = b$.

(ii) If A, G and H be respectively the arithmetic mean, the geometric mean and Harmonic mean of n positive integers $a_1, a_2, ..., a_n$ then $A \geq G \geq H$. The equality sign holds if and only if $a_1 = a_2 = ... = a_n$.

22. (i) If x is real and $Ax^2 + Bx + C = 0 \Leftrightarrow B^2 - 4AC \geq 0$.

(ii) If $A > 0$ and x is real then $Ax^2 + Bx + C \geq 0$

$$\Rightarrow \quad 4AC - B^2 > 0$$

23. **Triangle Inequality** : $|a| - |b| \leq |a + b| \leq |a| + |b|$.

24. $|\sum_i a_i| \leq \sum_i |a_i|$.

25. The lengths a, b, c can represent the sides of a triangle if and only if $a+b>c, b+c>a, a+c>b.$

26. **Weierstras's Inequality** : For positive number $a_1, a_2, a_3, ..., a_n$

$$(1+a_1)(1+a_2)(1+a_3)...(1+a_n) > 1+a_1+a_2+c_3+...+a_n$$

If a_i are fractions (less than one) then

$$(1-a_1)(1-a_2)...(1-a_n) > 1-(a_1+a_2+a_3+...+a_n)$$

27. **Cauchy Schwartz Inequality** : $(a_1b_1+a_2b_2+...+a_nb_n)^2 \le (a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2).$

28. **Tchebychev Inequality** : If $x_1 \ge x_2 \ge x_3 \ge ... x_n$ and $y_1 \ge y_2 \ge y_3 \ge ... y_n$ or $x_1 \le x_2 \le x_3 \le ... x_n$ and $y_1 \le y_2 \le y_3 \le ... y_n$. then

$$\frac{x_1y_1+x_2y_2+...+x_ny_n}{n} \ge \left(\frac{x_1+x_2+...+x_n}{n}\right)\left(\frac{y_1+y_2+...+y_n}{n}\right)$$

If one of the sequences is increasing and other is decreasing then the direction of the inequality changes.

29. If $a=\{a_1, a_2, ..., a_n\}$ are positive numbers, $b=\{b_i\}$ are various permutations of a_i then $\sum_{i=1}^{n} a_i^2 \ge \sum_{i=1}^{n} a_ib_i$ *i.e.,* $a_1^2+a_2^2+a_3^3+\ldots+a_n^2 \ge a_1a_2+a_2a_3+a_1a_3+\ldots+a_1a_n.$

30. The product $a^kb^lc^md^n$ when $a+b+c+d=Z$ will be maximum when $\left(\frac{a}{k}\right)^k\left(\frac{b}{l}\right)^l\left(\frac{c}{m}\right)^m\left(\frac{d}{n}\right)^n$ attains maximum.

Also using $AM > GM$

$$a^kb^lc^md^n \le \left(\frac{z}{k+l+m+n}\right)^{k+l+m+n} k^kl^lm^mn^n.$$

31. **Holder's Inequality** : $(a_1b_1+a_2b_2+...+a_nb_n)^{pq} \le (a_1^p+a_2^p+...+a_n^p)^q(b_1^q+b_2^q+...+b_n^q)^p$

Where $\frac{1}{P}+\frac{1}{q}=1,$ a_i and b_i are non negative real numbers.

Example 1. For any three positive real numbers *a, b,* and *c,* show that $a^2+b^2+c^2 \ge ab+bc+ca.$

Solution : We know that

$$\sqrt{ab} \le \frac{a+b}{2}$$

by squaring we have

$$ab \le \frac{a^2+b^2+2ab}{4}$$

$$ab-\frac{2ab}{4} \le \frac{a^2+b^2}{4}$$

$$\frac{2ab}{4} \le \frac{a^2+b^2}{4}$$

$$ab \le \frac{a^2+b^2}{2}$$

Similarly $$bc \le \frac{b^2+c^2}{2},\ ca \le \frac{c^2+a^2}{2}$$

Adding these three relations, we get

$$ab+bc+ca \le \frac{a^2+b^2}{2} + \frac{c^2+b^2}{2} + \frac{c^2+a^2}{2}$$

$$ab+bc+ca \le \frac{2(a^2+b^2+c^2)}{2}.$$ **Hence proved.**

Example 2. Show that $x^4 + y^4 + z^4 \ge x^2y^2 + x^2z^2 + y^2z^2$.

Solution: Let the three numbers be x^4, y^4 and z^4.

Applying A.M.,G.M. inequality to x^4 and y^4.

$$\frac{x^4+y^4}{2} \ge \sqrt{x^4y^4}$$

$\Rightarrow$ $$x^4 + y^4 \ge 2x^2y^2 \quad \ldots(1)$$

Similarly, $$x^4 + z^4 \ge 2x^2z^2 \quad \ldots(2)$$

And $$y^4 + z^4 \ge 2y^2z^2 \quad \ldots(3)$$

Adding (1), (2) and (3), we get

$$x^4 + y^4 + z^4 \ge x^2y^2 + x^2z^2 + y^2z^2.$$ **Hence proved.**

Example 3. If $x^2 + y^2 + z^2 = 27$ show that $x^3 + y^3 + z^3 \ge 81$.

Solution : Applying Cauchy-Schwartz inequality, to 2 sets

$$(x^{3/2}, y^{3/2}, z^{3/2}) \text{ and } (x^{1/2}, y^{1/2}, z^{1/2}),$$

We have $$(x^2 + y^2 + z^2)^2 \le (x^3 + y^3 + z^3)(x + y + z) \quad \ldots(1)$$

Again applying *C-S* inequality, to (x, y, z) and $(1,1,1)$.

We have $$(x + y + z)^2 \le 3(x^2 + y^2 + z^2) \quad \ldots(2)$$

Squaring both sides of (1) we have

$$(x^2 + y^2 + z^2)^4 \le (x^3 + y^3 + z^3)^2 (x + y + z)^2$$

On using (2), the above inequality yields,

$$(x^2 + y^2 + z^2)^4 \le 3(x^3 + y^3 + z^3)^2 (x^2 + y^2 + z^2) \quad \ldots(3)$$

Since $x^2 + y^2 + z^2 = 27,$ from (3), we get

$$(x^3 + y^3 + z^3)^2 \geq (81)^2$$

Taking +ve square root,

$\Rightarrow$ $$x^3 + y^3 + z^3 \geq 81$$ **Hence Proved.**

Example 4. Show that $(px + qy)(pq + xy) \geq 4pxqy;$ given p, q, x, y are +ve.

Solution : Given p, q, x, y are +ve

$\therefore$ px, qy, pq, xy are +ve numbers

Applying A.M., G.M. inequality to px and qy.

$$\frac{px + qy}{2} \geq \sqrt{pxqy}$$

$\Rightarrow$ $$px + qy \geq 2\sqrt{pxqy} \qquad \ldots(1)$$

Again, Applying A.M.– G.M. inequality to pq and xy

$\Rightarrow$ $$\frac{pq + xy}{2} \geq \sqrt{pxqy}$$

$\Rightarrow$ $$pq + xy \geq 2\sqrt{pxqy} \qquad \ldots(2)$$

Multiplying (1) and (2) we get

$$(px + qy)(pq + xy) \geq 4pxqy.$$

Example 5. Show that $a^3 + b^3 + c^3 \geq 3abc.$

Solution : If a^3, b^3 and c^3 we get

Applying A.M. – G.M. inequality.

$$\frac{a^3 + b^3 + c^3}{3} \geq \sqrt[3]{a^3 b^3 c^3}$$

$\Rightarrow$ $$a^3 + b^3 + c^3 \geq 3abc.$$ **Hence Proved**

Example 6. If a, b, c are unequal and positive show that $a^2 + b^2 + c^2 > ab + bc + ca.$

Solution : a, b, c are unequal and positive.

Applying A.M. – G.M. inequality to a^2 and b^2.

$$\frac{a^2 + b^2}{2} \geq \sqrt{a^2 b^2}$$

$\Rightarrow$ $$a^2 + b^2 > 2ab \qquad \ldots(1)$$

Similarly, $$b^2 + c^2 > 2bc \qquad \ldots(2)$$

$$c^2 + a^2 > 2ca \qquad \ldots(3)$$

Add to get $a^2+b^2+c^2>ab+bc+ca$

Example 7. If a be a positive number, prove that $a+\frac{1}{a}\geq 2.$

Solution : Let two positive numbers be a and $\frac{1}{a}$.

By A.M., G.M. Inequality

$$\frac{a+\frac{1}{a}}{2}\geq\sqrt{a\times\frac{1}{a}}$$

$\Rightarrow$ $a+\frac{1}{a}\geq 2.$ H.P.

Example 8. Prove that : $\frac{x^2}{1+x^4}\leq\frac{1}{2}$

or $\frac{1+x^2}{x^2}\geq 2$

or $\frac{1}{x^2}+x^2\geq 2.$

Solution : Let the 2 positive numbers be x^2 and $\frac{1}{x^2}$.

$\therefore$ By A.M. – G.M. Inequality,

$$\frac{x^2+\frac{1}{x^2}}{2}\geq\sqrt{x^2\times\frac{1}{x^2}}$$

$\Rightarrow$ $x^2+\frac{1}{x^2}\geq 2$

$\Rightarrow$ $\frac{x^4+1}{x^2}\geq 2$

$\Rightarrow$ $\frac{x^2}{1+x^4}\leq\frac{1}{2}.$ **Hence Proved**

Example 9. Proved that $\tan\alpha+\cot\alpha\geq 2$.

Solution : Applying A.M. – G.M. inequality to $\tan\alpha+\frac{1}{\tan\alpha}$

$$\frac{\tan\alpha+\frac{1}{\tan\alpha}}{2}\geq\sqrt{\tan\alpha\times\frac{1}{\tan\alpha}}\qquad\text{(By A.M.}\geq\text{G.M.)}$$

$$\tan\alpha+\cot\alpha\geq 2$$

Example 10. Which is bigger 17^{14} or 31^{11}.

Solution : Now $17^{14} > 16^{14}$

$16^{14} = (2^4)^{14}$

$\Rightarrow$ $17^{14} > 2^{56} > 2^{55}$

$2^{55} = (2^5)^{11} = (32)^{11}$

$\Rightarrow$ $17^{14} > 32^{11} > 31^{11}$

Thus $17^{14} > 31^{11}$

Example 11. Sow that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 5.

Solution : $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99} = (1^{99} + 4^{99}) + (2^{99} + 3^{99}) + ... + 5^{99}.$ …(1)

$$1^{99} + 4^{99} = (1+4)(1^{98} + 1^{97}.4 + 1^{96}.4^2 + ... + 4^{98}) = 5 \times p$$

Similarly, $2^{99} + 3^{99} = 5 \times q$

and above $5^{99} = 5 \times q$

thus the expression is divisible (1) is divisible by 5

Example 12. For positive a, b, c prove that.

$$a^3b + b^3c + c^3a \geq abc(a+b+c)$$

Solution : Let the two sets of numbers be

$$A = \{a, b, c\}$$

$$B = \{a^2b, b^2c, c^2a\}$$

By Tchebycheff inequality

$$a\frac{(a^2b) + b(b^2c) + c(c^2a)}{3} \geq \frac{(a+b+c)}{3}\left(\frac{a^2b + b^2c + c^2a}{3}\right)$$

$$3(a^3b + b^3c + c^3a) \geq (a+b+c)(a^2b + b^2c + c^2a) \quad \text{…(1)}$$

by A.G.–G.M. Inequality

$$\frac{a^2b + b^2c + c^2a}{3} \geq \sqrt[3]{a^3b^3c^3}$$

$$a^2b + b^2c + c^2a \geq 3abc \quad \text{…(2)}$$

$\therefore$ Hence from (1) and (2), we get

$$a^3b + b^3c + c^3a \geq (a+b+c)(abc).$$

Example 13. For positive *a*, *b*, *c* prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

Solution : As *a*, *b*, *c* are positive.

We may assume that $a+b\geq b+c\geq c+a$

Hence $$\frac{1}{a+b}\leq\frac{1}{b+c}\leq\frac{1}{c+a}$$

Let $$A=\{a+b, b+c, c+a\}$$

$$B=\left\{\frac{1}{a+b},\frac{1}{b+c},\frac{1}{c+a}\right\}.$$

Now apply Thebycheff's inequality

$$3\left\{\frac{a+b}{a+b}+\frac{b+c}{b+c}+\frac{c+a}{c+a}\right\}\leq\{(a+b)+(b+c)+(c+a)\}\times\left\{\frac{1}{(a+b)}+\frac{1}{(b+c)}+\frac{1}{(c+a)}\right\}$$

$$\Rightarrow \quad 3\times 3\leq 2(a+b+c)\left\{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right\}$$

$$\Rightarrow \quad \frac{a+b+c}{a+b}+\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}\geq\frac{9}{2}$$

$$\Rightarrow \quad \frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\geq\frac{9}{2}-3$$

$$\Rightarrow \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

Example 14. For positive *a*, *b*, *c* such that $abc=1$ show that $a^{b+c}b^{c+a}c^{a+b}\leq 1$.

Solution : From the given expression

$$(a)^{b+c}(b)^{c+a}(c)^{a+b}=a^{b+c}\left(\frac{1}{ac}\right)^{c+a}c^{a+b} \qquad [\because abc=1]$$

$$=a^{b+c}\cdot a^{-(c+a)}\cdot c^{-(c+a)}\cdot c^{a+b}$$

$$=\frac{a^{b+c-c-a}}{c^{c+a-a-b}}$$

$$=\frac{a^{b-a}}{c^{c-b}}.$$

We can assume that $a<b<c$ as the equation is symmetric.

So the numbers are positive integers raised to positive powers.

So the denominator is greater than the numerator.

Hence the above expression is ≤ 1.

i.e., $a^{b+c}\cdot b^{c+a}\cdot c^{a+b}\leq 1$

Example 15. Detemine all real number satisfying the inequality,

$$\frac{1}{2}\log(2x-1)+\log\sqrt{(x-9)}>1.$$

Solution :

$$\frac{1}{2}\log(2x-1)+\log\sqrt{(x-9)}>1$$

$\Rightarrow$ $$\log\sqrt{(2x-1)}+\log\sqrt{x-9}>1$$

$\Rightarrow$ $$\log[(2x-1)(x-9)]^{1/2}>\log 10$$

$\Rightarrow$ $$[(2x-1)(x-9)]^{1/2\times 2}>(10)^2$$

$\Rightarrow$ $$2x^2-18x-x+9>100$$

$\Rightarrow$ $$2x^2-18x-x-91>0$$

$\Rightarrow$ $$2x^2-19x-91>0$$

$\therefore$ $$\frac{-7}{2}<x<13$$

But for log to be defined

$$2x-1>0 \text{ and } x-9>0$$

$\Rightarrow$ $$x>\frac{1}{2} \text{ and } x>9.$$

i.e., $$9<x<13.$$

$\therefore$ Hence the required value $9<x<13$ Ans.

Example 16. If a, b, c, d are positive real numbers then show that $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 16$.

Solution : Without loss of generality, $a\leq b\leq c\leq d$

$\Rightarrow$ $$\frac{1}{a}\geq\frac{1}{b}\geq\frac{1}{c}\geq\frac{1}{d}.$$

Applying Tchebycheff's inequality

$$\left(\frac{a+b+c+d}{4}\right)\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)}{4}\geq\frac{\left(\frac{a}{a}+\frac{b}{b}+\frac{c}{c}+\frac{d}{d}\right)}{4}$$

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 4\times 4$$

or $$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 16.$$

Example 17. If a, b, c, d are +ve, prove that

$$a^5+b^5+c^5+d^5\geq abcd(a+b+c+d).$$

Solution : We choose two sets (a^4, b^4, c^4, d^4) and (a, b, c, d)

Applying inequality

$$4(a^5+b^5+c^5+d^5) \geq (a^4+b^4+c^4+d^4)(a+b+c+d)$$

Applying A.M. – G.M. inequality to a^4, b^4, c^4, d^4

$$\frac{a^4+b^4+c^4+d^4}{4} \geq \sqrt[4]{a^4b^4c^4d^4}$$

$$\Rightarrow \quad a^4+b^4+c^4+d^4 \geq 4abcd$$

$$\therefore \; a^5+b^5+c^5+d^5 \geq abcd(a+b+c+d).$$

Example 18. Show that $(n+1)^n \geq 2.4.6...\,2n.$

Solution : Consider the numbers $2.4.6...\,2n.$

$$\text{A.M.} \geq \text{G.M.}$$

$$\therefore \quad \frac{2+4+6+...+2n}{n} \geq \sqrt[n]{2\;4\;6\,...\,2n}$$

$$\therefore \quad \frac{\frac{n}{2}[4+2n-2]}{n} \geq \sqrt[n]{2\;4\;6\,...\,2n}$$

$$(n+1) > \sqrt[n]{2\;4\;6\,...\,2n}$$

$$(n+1)^n > 2.4.\,6...\,2n.$$

Example 19. For positive numbers x, y, z Show that

$$(x+y+z)^3 \geq 27(y+z-x)(z+x-y)(x+y-z)$$

Solution : Consider the numbers :

$$y+z-x,\; z+x-y \text{ and } x+y-z$$

$$\therefore \quad \left(\frac{x+y+z}{3}\right) \geq \sqrt[3]{(y+z-x)(z+x-y)(x+y-z)}$$

$$\therefore \quad \frac{(x+y+z)^3}{27} \geq (y+z-x)(z+x-y)(x+y-z)$$

$$(x+y+z)^3 \geq 27(y+z-x)(z+x-y)(x+y-z)$$

Example 20. Show that $\frac{x_1}{x_2}+\frac{x_2}{x_3}+...+\frac{x_{n-1}}{x_n}+\frac{x_n}{x_1} \geq n$ if $x_1, x_2, \dots, x_n$ are positive.

Solution : Consider the numbers:

$$\frac{x_1}{x_2}, \frac{x_2}{x_3}, ..., \frac{x_{n-1}}{x_n}, \frac{x_n}{x}$$

$\therefore \qquad \text{A.M.} \geq \text{A.M.}$

$$\therefore \quad \frac{\frac{x_1}{x_2}+\frac{x_2}{x_3}+\ldots+\frac{x_{n-1}}{x_n}+\frac{x_n}{x_1}}{n} \geq \sqrt{\frac{x_1}{x_2}, \frac{x_2}{x_3}, \frac{x_3}{x_4}, \ldots, \frac{x_{n-1}}{x_n}.\frac{x_n}{x}}$$

$$\therefore \ \frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_4}+\ldots+\frac{x_{n-1}}{x_n}+\frac{x_n}{x_1} \geq n.$$

Example 21. Show that $n^n > 1.3, \ldots, (2n-1)$.

Solution : Consider the numbers $1, 3, 5, \ldots, (2n-1)$.

$$\frac{1+3+5+\ldots+(2n-1)}{n} \geq \sqrt[n]{1.3.5\ldots(2n-1)}$$

$$\therefore \quad \frac{\frac{n}{2}[2+2n-2]}{n} \geq \sqrt[n]{1.3.5\ldots(2n-1)}$$

$$(n)^n \geq 1, 3, 5, \ldots, (2n-1)$$

Example 22. Find the minimum value of $2x+y$ subject to the condition $xy=8$, x and $y \in R_+$.

Solution : Applying A.M. – G.M. inequality

$$\frac{2x+y}{2} \geq \sqrt{2x.y}$$

$$\frac{2x+y}{2} \geq \sqrt{2.8}$$

$$\frac{2x+y}{2} \geq 4$$

$$2x+y \geq 8$$

$\therefore$ Minimum value of $2x+y=8$.

Example 23. Let a, b, c, d be real numbers such that $a<b<c<d$.

Prove that $(a+b+c+d)^2 > 8(ac+bd)$

Solution : Consider the quadratic polynomial with real coefficients

$f(x)=(x-a)(x-c)+(x-b)(x-d)$ or $f(x)=2x^2-(a+b+c+d)x+(ac+bd)$

since $a<b < c<d$, $f(a)>0$, $f(b)<0$, $f(c)<0$, $f(d)>0$

Hence $f(x)=0$ has real root between a and b and also between c and d (by Descarte's rule of signs) *i.e.*, the quadratic equation $f(x)=0$ has roots which are real and distinct.

$\therefore$ Discriminant is positive

i.e., $\qquad (a+b+c+d)^2 > 4\times 2(ac+bd)$

i.e., $\qquad (a+b+c+d)^2 > 8(ac+cd)$

Example 24. Without using tables, prove that

$$\frac{1}{\log_2 \pi}+\frac{1}{\log_5 \pi}>2$$

Solution : Let $\log_2 \pi = a$ and $\log_5 \pi = b$

$\therefore$ $2^a = \pi$ and and $5^b = \pi$

$\therefore$ $2=\pi^{1/a}$ and $5=\pi^{1/b}$

$\therefore$ $2\times 5=\pi^{\frac{1}{a}+\frac{1}{b}}$

i.e., $10=\pi^{\frac{1}{a}+\frac{1}{b}}$

But $10>\pi^2$. (Since $\pi = \frac{22}{7}$)

$\therefore$ $\pi^2<\pi^{\frac{1}{a}+\frac{1}{b}}$

$\therefore$ $2<\frac{1}{a}+\frac{1}{b}$

or $\frac{1}{a}+\frac{1}{b}>2$

Show that $\frac{1}{\log_2 \pi}+\frac{1}{\log_\pi 2}>2$

Example 25. If a, b, c are three numbers >0,

such that $a+b+c=1$, prove that

$$ab+bc+ca\le\frac{1}{3}$$

Solution : Now $\left.\begin{array}{l} a^2+b^2\ge 2ab \\ b^2+c^2\ge 2bc \\ c^2+a^2\ge 2ca \end{array}\right\}$

$\therefore$ $a^2+b^2+c^2\ge ab+bc+ca$ …(1)

$\therefore$ $(a+b+c)^2-2(ab+bc+ca)\ge ab+bc+ca$

i.e., $1\ge 3(ab+bc+ca)$

$\therefore$ $ab+bc+ca\le\frac{1}{3}$.

Example 26. Let a, b, c be real numbers with $0<a<1, 0<b<1, 0<c<1$ and $a+b+c=2$.

Prove that $\frac{a}{1-a}\cdot\frac{b}{1-b}\cdot\frac{c}{1-c}\ge 8$.

Solution : If x, y, z are positive, then

$$x+y \geq 2\sqrt{xy},\ y+z \geq 2\sqrt{yz} \quad z+x \geq 2\sqrt{zx}$$

$\therefore \quad (x+y)(y+z)(z+x) \geq 8xyz$ …(1)

Thus $8(1-a)(1-b)(1-c) \leq [(1-a+1-b)(1-b+1-c)(1-c+1-a)]$

$$8(1-a)(1-b)(1-c) \leq (2-a-b)(2-b-c)(2-c-a)$$

$$\leq cab$$

$\therefore \quad \dfrac{a}{1-a}.\dfrac{b}{1-b}.\dfrac{c}{1-c} \geq 8$

Example 27. Show that, for a triangle with radii of circum circle and incircle equal to R and r respectively, the inequality $R \geq 2r$ hols.

Solution : $R = \dfrac{abc}{4\Delta}$ and $r = \dfrac{\Delta}{s}$

$\therefore \quad \dfrac{R}{r} = \dfrac{abcs}{4\Delta^2} = \dfrac{abcs}{4.s(s-a)(s-b)(s-c)}$

$= \dfrac{2abc}{(b+c-a)(c+a-b)(a+b-c)}$ …(1)

Now, applying A.M. – G.M. inequality.

$$\frac{(b+c-a)+(c+a-b)}{2} \geq \sqrt{(b+c-a)(c+a-b)}$$

i.e., $c \geq \sqrt{(b+c-a)(c+a-b)}$ …(2)

similarly, $b \geq \sqrt{(c+b-a)(a+b-c)}$ …(3)

$a \geq \sqrt{(c+a-b)(a+b-c)}$ …(4)

$\therefore \quad abc \geq (b+c-a)(c+a-b)(a+b-c)$ …(5)

$\therefore \quad \dfrac{abc}{(b+c-a)(c+a-b)(a+b-c)} \geq 1$ …(6)

$\therefore \quad \dfrac{2abc}{(b+c-a)(c+a-b)(c+b-c)} \geq 2$ …(7)

i.e., $\dfrac{R}{r} \geq 2$ or $R \geq 2r$ …(8)

Example 28. Prove that $1 < \dfrac{1}{1001} + \dfrac{1}{1002} + ... + \dfrac{1}{3001} < 1\dfrac{1}{3}$.

Solution : Let $s = \dfrac{1}{1001} + ... + \dfrac{1}{3001} = \left(\dfrac{1}{1001} + \dfrac{1}{3001}\right) + ... + \left(\dfrac{1}{2000} + \dfrac{1}{2002}\right) + \dfrac{1}{2001}$

For any n, $n^2 - 4002n + 2001^2 \geq 0$ and hence $n(4002-n) \leq 2001^2$

$$s > 4002\left[\underbrace{\frac{1}{2001^2}+\ldots+\frac{1}{2001^2}}_{1000 \text{ times}}\right]+\frac{1}{2001}$$

$$= 4002\frac{1000}{2001^2}+\frac{1}{2001}$$

$$= \frac{2000+1}{2001}$$

$$= 1$$

Again, rate, terms hundred at *a* time

$$S = \frac{1}{1001}+\ldots+\frac{1}{3001} < 100\left(\frac{1}{1001}+\frac{1}{1101}+\ldots+\frac{1}{2901}\right)$$

$$< 100\left(\frac{1}{1000}+\frac{1}{1100}+\ldots+\frac{1}{2900}\right)$$

$$< \frac{1}{10}+\frac{1}{11}+..+\frac{1}{29}$$

$$< 5\left(\frac{1}{10}+\frac{1}{15}+\frac{1}{20}+\frac{1}{25}\right)$$

$$= \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{77}{60} < 1\frac{1}{3}$$

Hence $\quad 1 < s < 1\frac{1}{3}$

Example 29. If $abcd = 1$ show that $(1+a)(1+b)(1+c)(1+d) \geq 16.$

We expand $(1+a)1+b)(1+c)(1+d)$ and collect the terms in pairs such that the product of the terms in each pair is 1. Thus,

$$(1+a)(1+b)(1+c)(1+d) = (1+abcd)+\sum_4 (a+bcd)+\sum_3 (ab+cd)$$

where the integer written under the (symbol) Σ denotes the number of terms governed by Σ Now, each of the terms

$$(1+abcd), (a+bcd) \text{ and } (ab+cd)$$

is of the firm $x+\frac{1}{x}$ (with $x > 0$) and so it is ≥ 2.

Since there are $1+4+3=8$ such terms, we have the sum $\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right) \geq 8$

$$\sum_4 (a+bcd) = (a+bcd)+(b+acd)+(c+abd)+(d+abc)$$

$$\sum_3 (ab+cd) = (ab+cd)+(ac+bd)+(bc+ad)$$

Example 30. Show that $x \geq 0$ then $3x^3 = 6x^2 + 4 \geq 0$.

Solution : Since $3x^3 + 4 = 2x^3 + x^3 + 4$,

applying A.M. – G.M. inequality,

$$2x^3 + x^3 + 4 \geq 3.\sqrt[3]{2x^3 x^3 .4} = 3.2x^2 = 6x^2$$

Thus $$3x^3 - 6x^2 + 4 \geq 0$$

Example 31. Show that if x, y, z are non-negative reals, such that $x + y + z = 1$.

Solution : $x + y + z = 1$, then $\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right)\left(\frac{1}{z} - 1\right) \geq 8$

$$\frac{1}{x} - 1 = \frac{1 - x}{x} = \frac{(x + y + z) - x}{x} = \frac{y + z}{x}$$

also $$y + z \geq 2\sqrt{yz}$$

$$\therefore \quad (y + z)(z + x)(x + y) \geq 8\sqrt{yz}\sqrt{zx}\sqrt{xy}$$

$$\geq 8xyz$$

$$\therefore \quad \frac{y + z}{x}.\frac{z + x}{y}.\frac{x + y}{z} \geq 8$$

Example 32. If a, b, c, d are positive real numbers. Show that

$$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16.$$

Solution : $\frac{a + b + c + d}{4} \geq (abcd)^{1/4}$ and $\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}{4} \geq \left(\frac{1}{a}.\frac{1}{b}.\frac{1}{c}\frac{1}{d}\right)^{1/4}$

$\therefore$ Multiplying LHS and RHS respectively.

$$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16.$$

Example 33. If a, b, c are positive real numbers, prove that, $\frac{b^2 + c^2}{b + a} + \frac{c^2 + a^2}{c + a} + \frac{a^2 + b^2}{a + b} \geq (a + b + c)$

Solution : Now $$b^2 + c^2 \geq \frac{1}{2}(b + c)^2 \qquad \text{…(i)}$$

$$\because \quad \left(\frac{b + c}{2}\right)^2 \leq \frac{b^2 + c^2}{2}$$

$$\therefore \quad \frac{b^2 + c^2}{b + c} \geq \frac{1}{2}(b + c) \qquad \text{…(ii)}$$

Similarly $$\frac{c^2 + a^2}{c + a} \geq \frac{1}{2}(c + a) \text{ and } \frac{a^2 + b^2}{a + b} \geq \frac{1}{2}(a + b) \qquad \text{…(iii)}$$

Adding $\dfrac{b^2+c^2}{b+c}+\dfrac{c^2+a^2}{c+a}+\dfrac{a^2+b^2}{a+b}\geq\dfrac{1}{2}(2a+2b+2c)$

$$\geq(a+b+c).$$

Example 34. Show that $n^n\left(\dfrac{n+1}{2}\right)^{2n}>(n!)^3.$

Solution : Consider the unequal positive numbers $1^3, 2^3, 3^3, ..., n^3$.

.e., $$\frac{1^3+2^3+3^3+...+n^3}{n}>(1^3.2^3.3^3\,...\,n^3)^{1/n} \qquad \text{(A.M. } \geq \text{ G.M.)}$$

i.e., $$\frac{n^2(n+1)^2}{4n}>[(n!)^3]^{1/3}$$

Raising both sides to powers n,

$$n^n\left(\frac{(n+1)}{4}\right)^n>(n!)^3$$

i.e., $$n^n\left(\frac{n+1}{2}\right)^{2n}>(n!)^3$$

Example 35. If a, b, c are positive and $a+b+c=1$. Show that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\geq 9.$

Solution : On dividing by $(a+b+c)$ successively, by a, b, c we get

$$1+\frac{b}{a}+\frac{c}{a}=\frac{1}{a} \qquad \text{…(i)}$$

$$\frac{a}{b}+1+\frac{c}{b}=\frac{1}{b} \qquad \text{…(ii)}$$

$$\frac{a}{c}+\frac{b}{c}+1=\frac{1}{c} \qquad \text{…(iii)}$$

Adding, $$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)$$

$$\geq 3+2+2+2\geq 9$$

Example 36. Show that $\dfrac{1}{3}\leq\log_{20}3<\dfrac{1}{2}.$

Solution : $$9<20$$

$\therefore$ $$\log_{20}9<\log_{20}20$$

$\therefore$ $$\log_{20}(3)^2<1$$

i.e., $$2\log_{20}3<1$$

$\Rightarrow$ $$\log_{20}3<\frac{1}{2} \qquad \text{…(i)}$$

$$27 > 20$$

$\therefore$ $$\log_{20} 27 > \log_{20} 20$$

i.e., $$\log_{20}(3)^3 > 1$$

$\therefore$ $$3\log_{20} 3 > 1 \Rightarrow \log_{20} 3 > \frac{1}{3} \quad \ldots\text{(ii)}$$

Thus $$\frac{1}{3} < \log_{20} 3 < \frac{1}{2}.$$

Example 37. In ΔABC, Show that $\frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$

Solution : Now $$\frac{(a+b)+(b+c)+(c+a)}{3} \geq \{(a+b)(b+c)(c+a)\}^{1/3} \quad \ldots\text{(i)}$$

$$\frac{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}}{3} \geq \left\{\left(\frac{1}{a+b}\right)\left(\frac{1}{b+c}\right)\left(\frac{1}{c+a}\right)\right\}^{1/3} \quad \ldots\text{(ii)}$$

$\therefore$ Multiplying L.H.S. $(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{9}{2}$ …(iii)

i.e., $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{9}{2} - 3 = \frac{3}{2} \quad \ldots\text{(iv)}$$

Also $$b + c > a$$

$\therefore$ $$\frac{a}{b+c} < \frac{a+a}{a+b+c}$$

i.e., $$\frac{a}{b+c} < \frac{2a}{a+b+c}$$

$\therefore$ $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} < \frac{2(a+b+c)}{a+b+c} = 2$$

Thus $$\frac{3}{2} \leq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} < 2$$

Equality occurs when $a = b = c$.

Example 38. If *a*, *b*, *c* are sides of triangle show that $\left(1+\frac{b-c}{a}\right)^a \cdot \left(1+\frac{c-a}{b}\right)^b \cdot \left(1+\frac{a-b}{c}\right)^c < 1.$

Solution : Since a, b, c are the sides of a triangle,

$$a+b-c > 0, b+c-a > 0, c+a-b > 0 \quad \ldots\text{(i)}$$

Thus $1+\frac{b-c}{a}, 1+\frac{c-a}{b}, 1+\frac{a-b}{c}$ are all positive …(ii)

Take $1+\frac{b-c}{a}, 1+\frac{b-c}{a}$................. ‘*a*’ times

$$1+\frac{c-a}{b}, 1-\frac{c-a}{b} \text{................. } `b\text{' times}$$

$$1+\frac{a-b}{c}, 1-\frac{a-b}{c} \text{................. } `a\text{' times}$$

and apply A.M. – G.M. inequality

$\therefore$ $$a\left(1+\frac{b-c}{a}\right)+b\left(1+\frac{c-a}{b}\right)+c\left(1+\frac{a-b}{c}\right)$$

$$>\left[\left\{\left(1+\frac{b-c}{a}\right)\right\}^a \times\left(1+\frac{c-a}{b}\right)^b \times\left(1+\frac{a-b}{c}\right)^c\right]^{\frac{1}{c+b+c}}$$

LHS of the inequality $= 1$

i.e., $1 >$ R.H.S.

Thus $$\left(1+\frac{b-c}{a}\right)^a \times\left(1+\frac{c-a}{b}\right)^b \times\left(1+\frac{a-b}{c}\right)^c < 1.$$

Example 39. If a, b, c, d are four non-negative real numbers and $a+b+c+d=1$, show hat $ab+bc+cd \le \frac{1}{4}$.

Solution : $(a+b+c+d)^2 - 4(ab+bc+cd)$

$= a^2+b^2+c^2+d^2-2ab-2bc-2cd+2ac+2ad+2bd$

$= a^2-2ab+b^2+c^2+d^2-2cd-2bc+2ac+2ad+2bd$

$= (a-b)^2+(c-d)^2+2(a-b)(c-d)+4ad$

$= [(a-b)+(c-d)]^2+4ad \ge 0$ $(\because a, b, c, d \ge 0)$

$\Rightarrow$ $1-4(ab+bc+cd) \ge 0$

$\Rightarrow$ $4(ab+bc+cd) \le 1$

$\Rightarrow$ $(ab+bc+cd) \le \frac{1}{4}$

Aliter : the above problem can be solved by using A.M. – G.M. inequality we know that $(a+c)+(b+d)=1$.

$\Rightarrow$ $2\sqrt{(a+c)(b+d)} \le (a+c)+(b+d)$

$\Rightarrow$ $2\sqrt{(a+c)(b+d)} \le 1$

$\Rightarrow$ $4(a+c)(b+d) \le 1$

$\Rightarrow$ $ab+ad+bc+cd \le \frac{1}{4}$

$\Rightarrow \qquad ab+bc+cd \le \frac{1}{4} - ad.$

$\Rightarrow \qquad ab+bc+cd \le \frac{1}{4}. \qquad (\because ad \ge 0)$

Example 40. If a, b, c, d are positive, then prove that

$$(a^3b+b^3c+c^3d+d^3a)(ab^3+bc^3+cd^3+da^3) \ge 16(abcd)^2.$$

Solution : Applying Cauchy Schwarz inequality

$$a^3b = a_1^2, b^3c = a_2^2, c^3d = a_3^2, d^3a = a_4^2$$

and $\qquad ab^3 = b_1^2, bc^3 = b_2^2, cd^3 = b_3^2, da^3 = b_4^2,$

we get $\qquad a_1b_1 = a^2b^2, a_2b_2 = b^2c^2, a_3b_3 = c^2a^2, a_4b_4 = d^2a^2.$

Now applying A.M. – G.M. inequality, we get

$$(a^2b^2+b^2c^2+c^2d^2+d^2a^2)^2 \ge \left(4\sqrt[4]{a^4b^4c^4d^4}\right)^2 = 16a^2b^2c^2d^2$$

and hence $(a^3b+b^3c+c^3d+d^3a)(ab^3+bc^3+cd^3+da^3) \ge 16(abcd)^2.$

Example 41. Given that $x^2+y^2+z^2=8,$ prove that

$$x^3+y^3+z^3 \ge 16\sqrt{\frac{2}{3}}.$$

Solution : Applying Cauchy Schwartz inequality with

Let $\qquad x^{3/2}, y^{3/2}, z^{3/2}$ and $x^{1/2}, y^{1/2}, z^{1/2}$

we have $\qquad (x^2+y^2+z^2)^2 \le (x^3+y^3+z^3)(x+y+z)$

Again $\qquad x+y+z = x\times 1 + y\times 1 + z\times 1$

so $\qquad (x+y+z)^2 \le (x^2+y^2+z^2)(1^2+1^2+1^2)$

$$(x+y+z) \le \sqrt{3\times 8}$$

and hence $\qquad (x^3+y^3+z^3) \ge \frac{(x^2+y^2+z^2)^2}{(x+y+z)} = \frac{64}{\sqrt{3}.\sqrt{8}}$

$\Rightarrow \qquad x^3+y^3+z^3 \ge \frac{64}{4}.\frac{\sqrt{2}}{\sqrt{3}}$

$\Rightarrow \qquad x^3+y^3+z^3 \ge 16\sqrt{\frac{2}{3}}.$

Example 42. If $w^3+x^3+y^3+z^3=10,$ show that

$$w^4+x^4+y^4+z^4 \ge \sqrt[3]{2500}$$

Solution : Applying Cauchy Schwarz inequality for w^2, x^2, y^2, z^2 and $w, x, y, z,$ we get

$$(w^3 + x^3 + y^3 + z^3)^2 \le (w^4 + x^4 + y^4 + z^4)(w^2 + x^2 + y^2 + z^2) \quad \dots(1)$$

Again applying Cauchy Schwarz inequality with w^2, x^2, y^2, z^2 and 1, 1, 1, 1, we get

$$(w^2 + x^2 + y^2 + z^2)^2 \le (w^4 + x^4 + y^4 + z^4)^4$$

$$\Rightarrow \quad (w^2 + x^2 + y^2 + z^2) \le (w^4 + x^4 + y^4 + z^4)^2 \quad \dots(2)$$

$$\therefore \quad (w^4 + x^4 + y^4 + z^4) \ge \frac{(w^3 + x^3 + y^3 + z^3)^2}{(w^2 + x^2 + y^2 + z^2)}, \text{ by Eq. (1)}$$

$$\Rightarrow \quad \ge \frac{(w^2 + x^3 + y^3 + z^3)^2}{2(w^4 + x^4 + y^4 + z^4)^{1/2}}, \text{ by (2)}$$

$$\Rightarrow \quad (w^4 + x^4 + y^4 + z^4)^{3/2} \ge \frac{100}{2} = 50$$

$$\Rightarrow \quad w^4 + x^4 + y^4 + z^4 \ge 50^{2/3} \text{ or } \sqrt[3]{2500}$$

Example 43. If $(x+1)^2 > (5x-1)$ and $(x+1)^2 < (7x-3),$ find the integral values of x.

Solution : We have $(x+1)^2 > 5x-1$

$$\Rightarrow \quad (x+1)^2 - (5x-1) > 0$$

$$\Rightarrow \quad x^2 - 3x + 2 > 0$$

$$\Rightarrow \quad (x-2)(x-1) > 0$$

$$\Rightarrow \quad x > 2 \quad \dots(1)$$

$$\text{and} \quad x < 1 \quad \dots(2)$$

Again $(x+1)^2 < (7x-3)$

$$\Rightarrow \quad x^2 - 5x + 4 < 0$$

$$\Rightarrow \quad (x-4)(x-1) < 0$$

$$\Rightarrow \quad 1 < x < 4 \quad \dots(3)$$

Again Eq. (2) and Eq. (3) cannot satisfy one another.

Hence, we should consider Eq. (1) and Eq. (3) from which we get

$$x > 2 \text{ and } x < 4 \text{ so } x = 3.$$

Example 44. Prove that the polynomial

$$x^{9999} + x^{8888} + x^{7777} + \dots + x^{1111} + 1 \text{ is divisible by } x^9 + x^8 + x^7 + \dots + x + 1.$$

Solution : Let $P = x^{9999} + x^{8888} + x^{7777} + \dots + x^{1111} + 1$

and $Q = x^9 + x^8 + x^7 + \dots + x + 1$

$$P-Q = x^9(x^{9990}-1)+x^8(x^{8880}-1)\ +x^7(x^{7770}-1)+...+x(x^{1110}-1)$$

$$= x^9[(x^{10})^{999}-1]+x^8[(x^{10})^{888}-1]\ +\ x^7[(x^{10})^{777}-1]+...+x[(x^{10})^{111}-1] \qquad ...(1)$$

But $(x^{10})^n - 1$ is divisible by $x^{10}-1$ for all $n \geq 1$

$\therefore$ RHS of Eq. (1) is divisible by $x^{10}-1$.

$\therefore$ $P-Q$ is divisible by $x^{10}-1$ and hence divisible by $x^9+x^8+...+1$.

Example 45. Given that the equation $x^4+px^3+qx^2+rx+s=0$ has four positive roots, prove that (i) $pr-16s \geq 0$, (ii) $q^2-36s \geq 0$.

Solution : Let $\alpha, \beta, \gamma, \delta$ be he four positive roots of the given equation. Then,

$$\alpha+\beta+\gamma+\delta = -p \qquad ...(1)$$

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta g+\beta\gamma+\gamma\delta = q \qquad ...(2)$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r \qquad ...(3)$$

$$\alpha\beta\gamma\delta = s \qquad ...(4)$$

(i) Using A.M. – G.M. inequality in Eq. (1) and Eq. (3), we get

$$\frac{\alpha+\beta+\gamma+\delta}{4}.\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{4} \geq \sqrt[4]{\alpha\beta\gamma\delta}\sqrt[4]{\alpha^3\beta^3\gamma^3\delta^3}$$

$$= \alpha\beta\gamma\delta = s$$

$$\frac{-p}{4}.\left(\frac{-r}{4}\right) \geq s$$

$$pr \geq 16s \text{ or } pr-16s \geq 0$$

(ii) Applying A.M. – G.M. inequality in Eq. (2), we get

$$\frac{q}{6} \geq \sqrt[6]{\alpha^3\beta^3\gamma^3\delta^3} = \sqrt{s}$$

$$q^2 \geq 36s \text{ or } q^2-36s \geq 0.$$

Example 46. If a, b and c are positive real numbers such that $a+b+c=1$, prove that $(1+a)(1+b)(1+c) \geq 8(1-a)(1-b)(1-c)$.

Solution : We know $a+b+c=1$

$\Rightarrow$ $b+c = 1-a$

and $1+a = 1+1-(b+c) = (1-b)+(1-c)$

and since $a+b+c=1$ where a, b and c are positive real numbers, so $1-b$ and $1-c$ are positive.

Applying A.M. – G.M. inequality, we get

$$1+a = (1-b)+(1-c) \geq 2\sqrt{(1-b)(1-c)} \qquad ...(1)$$

$$1+b=(1-a)+(1-c)\geq 2\sqrt{(1-a)(1-c)} \qquad \ldots(2)$$

and $$1+c=(1-b)(1-a)\geq 2\sqrt{(1-b)(1-a)} \qquad \ldots(3)$$

Multiplying Eq. (1), Eq. (2) and Eq. (3), we get

$$(1+a)(1+b)(1+c)\geq 8(1-a)(1-b)(1-c).$$

Example 47. If a, b and c are the sides of a triangle and $a+b+c=2$, then prove that $a^2+b^2+c^2+2abc<2.$

Solution : We know $a+b+c=2$ and squaring, we get

$$4=(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

$\Rightarrow \quad a^2+b^2+c^2=2(2-ab-bc-ca)$

Adding $2abc$ if both sides, we get

$$a^2+b^2+c^2+2abc=2(2-ab-bc-ca+abc)$$

To prove $a^2=b \ \ +c^2+2abc<2,$ it is enough to prove that

$$2(2-ab-bc-ca+abc)<2$$

or $$2+abc-ab-bc-ca)<1$$

or $$ab+bc+ca-abc-1>0$$

$\because$ $$a+b+c=2s=2$$

$\Rightarrow$ $$s=1$$

Example 48. For $n\in N, n>1,$ show that

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n^2}>1.$$

Solution : We have

$$\underbrace{\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n^2}}_{(n^2-n)\text{ terms}}>\frac{1}{n}+\underbrace{\left(\frac{1}{n^2}+\frac{1}{n^2}+\frac{1}{n^2}+\ldots+\frac{1}{n^2}\right)}_{(n^2-n)\text{ terms}}$$

$$\Rightarrow \quad \frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n^2}>\frac{1}{n}+\frac{(n^2-n)}{n^2}=\frac{1}{n}+1-\frac{1}{n}=1.$$

Example 49. Let a, b, c be real numbers with $0<a, b, c<1$ and $a+b+c=2.$ Prove that

$$\frac{a}{1-a}.\frac{b}{1-b}.\frac{c}{1-c}\geq 8.$$

Solution : Here we use A.M. – G.M.

$$a=\frac{(a+b-c)+(a-b+c)}{2}\geq\sqrt{(a+b-c)(a-b+c)}$$

$$b = \frac{(b+a-c)+(b-a+c)}{2} \geq \sqrt{(b+a-c)(b-a+c)}$$

$$c = \frac{(c+a-b)+(c-a+b)}{2} \geq \sqrt{(c+a-b)(c-a+b)}$$

$\therefore$ $$a.b.c = \frac{[(a+b-c)+(a-b+c)][(b+a-c)+(b-a+c)][(c+a-b)+(c-a+b)]}{8}.$$

$$\geq \sqrt{(a+b-c)(a-b+c)(b+a-c)(b-a+c)(c+a-b)(c-a+b)}$$

$$= (2-2c)(2-2a)(2-2b)$$

$$= 8(1-c)(1-a)(1-b). \qquad [\because a+b+c=2]$$

$\therefore$ $$\frac{a}{1-a}.\frac{b}{1-b}\frac{c}{1-c} \geq 8.$$