16^{th} KVS Junior Mathematical Olympiad -2013

Time : 3 hrs.]

[M.M : 100

Note : *(i) All questions are compulsory.*
(ii) Each question carries 10 marks.

1. Prove that for every prime $p > 7$, $p^6 - 1$ is divisible by 504.

2. (a) Determine the smallest positive integer x, whose last digit is 6 and if we erase this 6 and put it in left most of the number so obtained, the number becomes 4x.

   (b) For any real numbers a and b, prove that :
   $$3a^4 - 4a^3b + b^4 \geq 0$$

3. If squares of the roots of $x^4 + bx^2 + cx + d = 0$ are α, β, γ and δ, then prove that :
   $$64\,\alpha\beta\gamma\delta - [\,4\sum\alpha\beta - (\sum\alpha)^2\,]^2 = 0$$

4. (a) Let a,b,c be the length of the sides of a triangle and r be the in-radius. Show that :
   $$r < \frac{a^2+b^2+c^2}{3(a+b+c)}$$

   (b) A family consists of a grand father, 6 sons & daughters and 4 grand children. They are to be seated in a row for a dinner. The grand children wish to occupy the two seats at each end and the grand father refuses to have a grand child on either side of him. Determine the number of ways in which they can be seated for the dinner.

5. A semi-circle is drawn outwardly on chord AB of the circle with centre O and unit radius. The perpendicular from O to AB, meets the semi-circle on AB at C. Determine the measure of $\angle AOB$ and length AB so that OC has maximum length.

6. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$,
   prove that $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

7. A point 'A' is randomly chosen in a square of side length 1 unit. Find the probability that the distance from A to the centre of the square does not exceed x.

8. (a) Two parallel chords in a circle have lengths 10 cm and 14 cm and distance between them is 6 cm. If a chord parallel to these chords and midway between them is of length $\sqrt{a}$, find the value of a.

   (b) The line joining two points A (2,0) and B (3,1) is rotated about point A in anti-clockwise direction through an angle of 15^0. Find the equation of the line in the new position. If B goes to C in the new position, find the coordinates of C.

9. Find the number of numbers $\leq 10^8$ which are neither perfect squares, nor perfect cubes, nor perfect fifth powers.

10. Let PQRS be a rectangle such that PQ = a and QR = b. Suppose r_1 is the radius of the circle passing through P and Q and touching RS and r_2 is the radius of the circle passing through Q and R and touching PS. Show that :
    $$5(a+b) \leq 8(r_1 + r_2)$$

17th KVS Junior Mathematical Olympiad -2014

Time-3 hours

M.M.100

NOTE : *All questions are compulsory. Each question carries 10 marks. Use of electronic gadgets is not allowed.*

1. Prove that for no integer n, $n^6+3n^5-5n^4-15n^3+4n^2+12n+3$ is a perfect square.

2. Two dice are thrown once simultaneously. Let E be the event "Sum of numbers appearing on the dice." What are the members of E ? Can you load these dice (not necessarily in the same way) such that all members of E are equally likely ? Give justification.

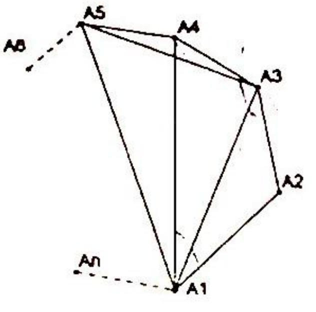
3. Let sin x + sin y = a and cos x+cos y = b, Show that $\tan\frac{x}{2}$ and $\tan\frac{y}{2}$ are two roots of the equation :

$$(a^2+b^2+2b)t^2-4at+(a^2+b^2-2b)=0$$

4. In a triangle ABC, angle A is twice the angle B . Show that $a^2 = b(b+c)$, where a, b and c are the sides opposite to angle A, B and C respectively.

5. A,B,C,D are four points on a circle with radius R such that AC is perpendicular to BD and meets BD at E. Prove that : $EA^2+EB^2+EC^2+ED^2=4R^2$.

6. Suppose $A_1A_2A_3$-------A_n is an n-sided regular polygon such that :



$$\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}.$$ Determine the number of sides of the polygon.

7. Find all positive integers a,b for which the number $\frac{\sqrt{2}+\sqrt{a}}{\sqrt{3}+\sqrt{b}}$ is a rational number.

8. If a,b,c are positive real numbers, prove that :

$$\frac{\sqrt{a+b+c}+\sqrt{a}}{b+c}+\frac{\sqrt{a+b+c}+\sqrt{b}}{c+a}+\frac{\sqrt{a+b+c}+\sqrt{c}}{a+b}\geq\frac{9+3\sqrt{3}}{2\sqrt{a+b+c}}$$

9. Find integers a and b such that x^2-x-1 divides $ax^{17}+bx^{16}+1=0$.

10. Consider the equation $x^4-18x^3+kx^2+174x-2015=0$. If the product of two of the four roots of the equation is -31, find the value of k.